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A finite population ESS and a long run equilibrium in an n players coordination game

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Abstract

Kandori, M., Mailath, G., Rob, R. [(1993), Learning, mutation, and long run equilibria in games, Econometrica 61, 29–56] (hereafter KMR) showed that a risk-dominant equilibrium is selected as a long run equilibrium in a symmetric 2×2 coordination game. But a risk-dominant equilibrium and a long run equilibrium exactly coincide only in the case of a large (infinite) population. In this paper we will show that $N/2$-stability of a finite population evolutionarily stable strategy defined by Schaffer, M.E. [1988. Evolutionarily stable stategies for a finite population and a variale contest size, Journal of Theoretical Biology 132, 469–478] is a necessary and sufficient condition for a long run equilibrium in the sense of KMR in an n ($2\leqq n\leqq N$) players coordination game with two alternative strategies for each player, where N denotes the population of players, which may be small.

Keywords: Finite population ESS; Long run equilibrium

JEL classification: C72

1. Introduction

Kandori et al. (1993) (hereafter KMR) studied Darwinian (or natural selection) dynamics with a random mutation in a 2×2 symmetric game. (A dynamic process is Darwinian if strategies that are more successful in the current period become more common in the next period.) In particular, they showed that, in a symmetric 2×2 coordination game, which has two Nash equilibria, a risk-dominant equilibrium is selected as a long run equilibrium, that is, it is selected in a limit of an invariant

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distribution as a mutation rate tends toward zero. (A risk-dominant equilibrium may be Pareto-dominated by another equilibrium. Risk-dominance is as defined by Harsanyi and Selten (1988).) But, as KMR's Corollary 3 suggested, a risk-dominant equilibrium and a long run equilibrium exactly coincide only in the case of a large population. In this paper we will show that $N/2$-stability of a finite population evolutionarily stable strategy defined by Schaffer (1988) is a necessary and sufficient condition for a long run equilibrium in the sense of KMR in an n ($2 \leqq n \leqq N$) players coordination game with two alternative strategies for each player, where N denotes the population of players, which may be small[1]. An example of an n person coordination game is a model of consumers' brand choice with *network externalities* as analyzed by Katz and Shapiro (1985) in which utility of consumers buying a good of one brand is increasing in the number of other consumers who buy and use that brand.

Schaffer (1988) proposed a concept of an evolutionarily stable strategy (ESS) for a finite (or small) population as a generalization of the standard ESS concept for an infinite (or large) population by Maynard-Smith (1982). We call it a *finite population ESS*. He showed that a finite population ESS is not generally a Nash equilibrium strategy[2]. In Schaffer (1989) he applied this concept to an economic game, and showed that a strategy which survives in economic natural selection is a strategy which maximizes a relative, not absolute, payoff. He assumed the following survival rule in economic natural selection. Players are born with strategies and can't change their strategies in response to changing circumstances. At the end of each period, if the payoff of Player i is larger than the payoff of Player j, the probability that Player i will survive in the next period is larger than the probability that Player j will survive in the next period.

Alternatively we can assume that the survival rule operates on strategies, not players, and the population proportion of successful strategies grows with players' imitative behavior. In this paper we consider the following model. Each player can observe decisions of other players, and his own and other players' payoffs, but does not know the whole payoff structure of the game, and can't compute his best response to other players' strategies. On the other hand, each player knows that other players have the same payoff function as his own, and that the game is symmetric. When all players choose the same strategy, denoted by s_1, in a symmetric game their payoffs are equal, and they do not have an opportunity to mimic another person and so can't change their strategies. Now, suppose that one player (a mutant player) chooses a different strategy, denoted by s_2. If the payoff of this player is higher (or lower) than the payoffs of other players (non-mutant players), the non-mutant players have (or the mutant player has) an incentive to change their strategies to s_2 (or s_1). In such a situation imitation of other players' successful strategies seems to be an appropriate assumption of the behavior of players[3].

[1]Kim (1996) studied a long run equilibrium in a coordination game. But he considers only a case of an infinite population.

[2]Rhode and Stegeman (1996) showed that a long run equilibrium need not be a Nash equilibrium in dominant strategy games for a small population.

[3]For a more detailed analysis of imitation behavior, see Schlag (1998).

Hansen and Samuelson (1988) also presented analyses of evolution in economic games. They showed that with a small number of players, a surviving strategy in economic natural selection, they called such a strategy a *universal survival strategy*, is not a Nash equilibrium strategy. Their *universal survival strategy* is essentially equivalent to Schaffer's finite population ESS. They said, ''In real-world competition, firms will be uncertain about the profit outcomes of alternative strategies. This presents an obvious obstacle to instantaneous optimization. Instead, firms must search for and learn about more profitable strategies. As Alchian (1950) emphasizes, an important mechanism for such a search depends on a comparison of observed profitability across the strategies used by market participants. That is, search for better strategies is based on *relative* profit comparisons.''

Some recent papers such as Robson and Vega-Redondo (1996) and Vega-Redondo (1997) considered a model of stochastic evolutionary dynamics assuming imitation dynamics of players' strategies. On the other hand, some other papers such as Kandori and Rob (1995, 1998) assumed best response dynamics. In best response dynamics each player chooses a strategy in period $t+1$ which is a best response to other players' strategies in period t. Thus the players must know the payoff structure of the game, and must be able to compute their best responses. In imitation dynamics, on the other hand, players simply mimic other players' successful strategies. We think that imitation dynamics are more appropriate than best response dynamics for an economic game with boundedly rational players. There are experimental studies about the role of imitation in economic decision making, such as Pingle and Day (1996) and Offerman and Sonnemans (1998). Pingle and Day (1996) argued that imitation plays an important role in real-world decision making, because it is one of the procedures that allows decision makers to economize on decision costs.

The model of this paper is as follows. n players are randomly chosen from the population, and play a symmetric coordination game many times in each period. Each player chooses one of two alternative strategies, s_1 and s_2. We will show the following result. If and only if s_1 (or s_2) is an $N/2$-stable finite population ESS, a state in which all players choose s_1 (or s_2) is a long run equilibrium.

In the next section we consider a finite population ESS. In Section 3 we examine a relationship between stability of a finite population ESS and a long run equilibrium.

2. A finite population ESS and its stability

Consider an n $(2 \leqq n \leqq N)$ players symmetric coordination game. N $(\geqq 2)$ denotes the population of players which may be small. n is called the contest size of the game. There are two alternative strategies for each player. Denote two strategies by s_1 and s_2. The payoff of the players who choose s_1 when k $(0<k<n)$ players choose s_1 and $n-k$ players choose s_2 by $a_1(k)$. Similarly, the payoff of the players who choose s_2 in the same case by $a_2(k)$. $a_1(k)$ and $a_2(k)$ are non negative, and satisfy the following assumptions,

Assumption 1. $a_1(k)$ is increasing in k, and $a_2(k)$ is decreasing in k.

and

Assumption 2. $a_1(n) > a_2(n-1)$ and $a_2(0) > a_1(1)$.

Assumption 1 means that as the number of the players who choose s_1 (or s_2) increases, the payoff of the players who choose s_1 (or s_2) increases. Assumption 2 implies that a state in which all players choose s_1 and a state in which all players choose s_2 are two strict Nash equilibria.

The players are randomly chosen from the population, and play the game many times in each period. A finite population ESS for the contest size n is defined as follows. Consider a state in which all players choose s_1. If, when one player (a mutant player) chooses a different strategy, s_2, the average payoff of the players who choose s_1 is larger than the average payoff of the mutant player, then s_1 is a finite population ESS. The condition for s_2 to be a finite population ESS is parallel.

If $n = N$, the game is a so called *playing the fields model*. An example of an n person coordination game is a model of consumers' brand choice, for example personal computer operating system, with *network externalities* as analyzed by Katz and Shapiro (1985), in which utility of consumers buying a good of one brand is increasing in the number of other consumers who buy and use that brand.

Denote the average payoff of the players who choose s_i, $i = 1, 2$, when z $(0 < z < N)$ players choose s_1, $N - z$ players choose s_2 and the contest size is n, by $\pi_i(n, z, N - z)$. And denote the difference between the average payoff of the players who choose s_1 and the average payoff of the players who choose s_2 by

$$\varphi(n, z, N - z) = \pi_1(n, z, N - z) - \pi_2(n, z, N - z).$$

Then, s_1 is a finite population ESS if [4]

$$\pi_1(n, N - 1, 1) > \pi_2(n, N - 1, 1) \text{ or } \varphi(n, N - 1, 1) > 0. \tag{1}$$

Similarly, s_2 is a finite population ESS if

$$\pi_2(n, 1, N - 1) > \pi_1(n, 1, N - 1) \text{ or } \varphi(n, 1, N - 1) < 0.$$

Now we will show

Lemma 1. (1) $\pi_1(n, z, N - z)$ *is increasing in* z. (2) $\pi_2(n, z, N - z)$ *is decreasing in* z.

Proof. See Appendix A.

This lemma means that the more the number of the players who choose s_1 (or s_2) is, the larger their payoff is, and the smaller the payoff of the players who choose s_2 (or s_1) is.

[4] Schaffer's original definition is weaker. He defines s_1 as a finite population ESS if (2.1) is satisfied with weak inequality. We adopt the definition with strong inequality. Concerning definitions of a finite population ESS, see Crawford (1991).

From Lemma 1

Lemma 2. *$\varphi(n, z, N-z)$ is increasing in z.*

Schaffer (1988) further defines stability of finite population ESSs. Consider a state in which all players choose s_1. If, when M $(0<M<N)$ or fewer mutant players choose s_2, the average payoff of the players who choose s_1 is larger than the average payoff of the mutant players, then s_1 is called an *M-stable* (finite population) ESS. Hereafter we abbreviate "finite population". Formally, s_1 is an M-stable ESS if

$$\varphi(n, N-M', M')>0 \text{ for } 0<M' \leqq M, \tag{2}$$

and

$$\varphi(n, N-M', M')<0 \text{ for } M'>M. \tag{3}$$

Similarly, s_2 is an M-stable ESS if

$$\varphi(n, M', N-M')<0 \text{ for } 0<M' \leqq M, \tag{4}$$

and

$$\varphi(n, M', N-M')>0 \text{ for } M'>M. \tag{5}$$

From Lemma 2 $\varphi(n, N-M', M')$ is increasing in $N-M'$, and decreasing in M'. Thus, if (2) holds for $M'=M$, it holds for any $M'<M$. And if (3) holds for $M'=M+1$, it holds for any $M'>M+1$. Therefore it is necessary and sufficient for s_1 to be an M-stable ESS that (2) holds for $M'=M$ and (3) holds for $M'=M+1$.

Similarly, it is necessary and sufficient for s_2 to be an M-stable ESS that (4) holds for $M'=M$ and (5) holds for $M'=M+1$.

When $M=N/2$ we obtain the condition for s_1 to be an $N/2$-stable ESS as follows[5],

$$\varphi\left(n, \frac{N}{2}, \frac{N}{2}\right)>0. \tag{6}$$

$N/2$-stability of s_1 means that, when $N/2$ (a half of the population) or fewer mutant players choose s_2, the average payoff of the players who choose s_1 is larger than the average payoff of the mutant players. Similarly, the condition for s_2 to be an $N/2$-stable ESS is

$$\varphi\left(n, \frac{N}{2}, \frac{N}{2}\right)<0.$$

If s_1 is M-stable, s_2 is $(N-M-1)$-stable. If s_1 is $(N-1)$-stable (globally stable), then s_2 is not a finite population ESS. If s_1 is $N/2$-stable, then s_2 can't be $N/2$-stable.

[5]Strictly speaking, (6) is the condition for s_1 to be a $N/2$ *or higher*-stable ESS. If the following condition holds in addition to (6), s_1 is exactly $N/2$-stable.

$$\varphi\left(n, \frac{N}{2}-1, \frac{N}{2}+1\right)<0.$$

3. The relation between stability of a finite population ESS and a long run equilibrium

KMR presented an analysis of long run equilibria of stochastic evolutionary dynamics. Our model is an extension to an n players game from their original model with a two players game. N players are repeatedly matched to play an n $(2 \leqq n \leqq N)$ players symmetric coordination game with two alternative strategies, which satisfies Assumptions 1 and 2, in each period. $z_t \in Z$, $Z \equiv \{0, 1, 2, \ldots, N\}$, denote the number of the players choosing s_1 in period t. The number of the players who choose s_1 in period $t+1$ is determined by deterministic and stochastic processes. The deterministic component is $z_{t+1} = b(z_t)$. $b(z)$ satisfies the following Darwinian property,

$$b(z) > z \text{ when } \varphi(n, z, N-z) > 0,$$

and

$$b(z) < z \text{ when } \varphi(n, z, N-z) < 0,$$

where

$$\varphi(n, z, N-z) = \pi_1(n, z, N-z) - \pi_2(n, z, N-z).$$

We assume $b(0) = 0$ and $b(N) = N$. $\pi_1(n, z, N-z)$ and $\pi_1(n, z, N-z)$ are the average payoffs of the players who choose, respectively, s_1 and s_2. They are the same as in the previous section. The stochastic component is a random mutation. In each period each player randomly switches his strategy with probability ε. The complete dynamic is represented by $z_{t+1} = b(z_t) + x_t$, where x_t is the net number of the players who change their strategies to s_1. z_t is a Markov process. Consider a limit of a stationary distribution of the Markov process z_t as $\varepsilon \to 0$. Long run equilibria are states which are assigned positive probability in a limit. We denote by z a state in which the number of the players who choose s_1 is z.

Now consider a z-tree, which is a directed graph consisting of directed branches (z_1, z_2) including a branch connected to z, z_2 is a successor of z_1, where, (1) except for z, each state has a unique successor, and, (2) there is no closed loop.[6]

KMR define v_z that is the number of mutations contained in a z-tree. Their Lemmas 1, 2, 3 and Theorem 1 showed that long run equilibria comprise states having minimum v_z.

Arranging KMR's Theorem 3 and the theorem in Rhode and Stegeman (1996) we will show

Theorem 1. (1) *If* $\varphi(n, N/2, N/2) > 0$, *a long run equilibrium is the state* $z = N$, *where all players choose* s_1. (2) *If* $\varphi(n, N/2, N/2) < 0$, *a long run equilibrium is the state* $z = 0$, *where all players choose* s_2.

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[6]For more details about a tree see KMR, Vega-Redondo (1996) and Vega-Redondo (1997).

Proof.

1. Since $\varphi(n, z, N-z)$ is increasing in z, we need no mutation to reach N from z', where z' is the smallest state such as $\varphi(n, z, N-z)>0$. Thus v_N is the number of mutations needed to reach z' from 0, and it is smaller than or equal to $N/2$. v_z for $z' \leqq z < N$ is larger than v_N, $(0 \to z', N \to z)$[7]. Since $\pi_2(n, z, N-z) - \pi_1(n, z, N-z) = -\varphi(n, z, N-z)$ is decreasing in z, we need no mutation to reach 0 from z'', where z'' is the largest state such as $\varphi(n, z, N-z)<0$. Thus v_0 is the number of mutations needed to reach z'' from N. Since $\varphi(n, N/2, N/2)>0$, z'' is smaller than $N/2$, and v_0 is larger than $N/2$. v_z for $0<z \leqq z''$ is larger than v_0, $(\mathrm{N} \to z'', 0 \to z)$[8].
   Therefore the state N is a long run equilibrium.
2. The proof of case (2) is parallel to the proof of case (1). □

If $\varphi(n, N/2, N/2)=0$, both 0 and N are long run equilibria.

From the arguments in the previous section we find that the condition in this theorem for s_1 (or s_2) to be a long run equilibrium strategy is the same as the condition for s_1 (or s_2) to be an $N/2$-stable ESS. Therefore we obtain

Theorem 2. *If and only if s_1 (or s_2) is an $N/2$-stable ESS, the state in which all players choose s_1 (or s_2) is a long run equilibrium.*

4. An example

As an example of the analysis in this paper we consider consumers' brand choice of a good with network externalities[9]. There are N ($N \geqq 0$) consumers who buy one unit of a good. There are two brands of the good, Brand 1 and Brand 2. We denote the number of the consumers who buy and use Brand i by y_i, for $i=1,2$. The game of the consumers is an N players game, that is, $n=N$ and the game is a playing the fields model. The utility of a consumer who buys and uses a unit of Brand i depends on the number of the consumers who buy and use a unit of that brand. Considering linear utility functions, we assume that the surplus of a consumer who buys a unit of Brand 1 is represented by

$$a_1(y_1) = \alpha y_1 - p_1,$$

and the surplus of a consumer who buys a unit of Brand 2 is written as

$$a_2(y_2) = \beta y_2 + \gamma - p_2 = \beta(N - y_1) + \gamma - p_2.$$

p_1 and p_2 are the prices of Brand 1 and Brand 2. α, β and γ are positive constants.

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[7]We need no mutation to reach z from z'.

[8]We need no mutation to reach z from z''.

[9]Kandori and Rob (1998) analyzed technology choice with network externalities in a coordination game. They considered a two players game with more than two strategies and best response dynamics. We consider an n players game with two strategies.

Then, $a_1(y_1)$ is increasing in y_1 and decreasing in y_2, and $a_2(y_2)$ is increasing in y_2 and decreasing in y_1.

If α, β and γ satisfy the following conditions, $a_1(y_1)$ and $a_2(y_2)$ satisfy Assumptions 1 and 2.

$$\alpha N - p_1 > \beta + \gamma - p_2, \tag{7}$$

and

$$\beta N + \gamma - p_2 > \alpha - p_1. \tag{8}$$

Now we assume $p_1 = p_2$, and $N = 6$. Then the condition for Brand 2 to be an $N/2$-stable ESS is

$$3\beta + \gamma > 3\alpha. \tag{9}$$

We define the risk dominance in an n players game as follows. If, for a consumer, when he has a conjecture that each other consumer chooses Brand 1 or 2 with probability 1/2, his expected surplus when he chooses Brand 1 is larger than his expected surplus when he chooses Brand 2, then Brand 1 is risk dominant. The condition for Brand 1 to be risk dominant is[10]

$$3.5\alpha > 3.5\beta + \gamma. \tag{10}$$

If we assume $\alpha = 6$, $\beta = 2$ and $\gamma = 13$, then (7), (8), (9) and (10) hold when $p_1 = p_2$ and $N = 6$. Thus the long run equilibrium is the state where all consumers buy and use Brand 2 although Brand 2 is risk dominated by Brand 1. In this case Brand 2 is Pareto dominated, as well as risk dominated, by Brand 1 since each consumer's utility in the equilibrium where all consumers buy Brand 2 is 25, and each consumer's utility in the equilibrium where all consumers buy Brand 1 is 36.

(10) means that if a half of the consumers other than one consumer (two and a half consumers) choose Brand 1 and 2, the surplus of that consumer when he buys Brand 1 is larger than his surplus when he buys Brand 2. (9), on the other hand, means that if a half of the consumers (three consumers) choose Brand 1 and 2, the surplus of the consumers buying Brand 2 is larger than the surplus of the consumers buying Brand 1. Thus three mutations are sufficient to upset the equilibrium where all consumers buy Brand 1, but we need at least four mutations to upset the equilibrium where all consumers buy Brand 2. Therefore Brand 2 is the long run equilibrium strategy.

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[10]The expected surplus of a consumer when he chooses Brand 1 is larger than his expected surplus when he chooses Brand 2 if

$$\frac{1}{32}[(6 + 5 \times 5 + 4 \times 10 + 3 \times 10 + 2 \times 5 + 1)\alpha] > \frac{1}{32}[(6 + 5 \times 5 + 4 \times 10 + 3 \times 10 + 2 \times 5 + 1)\beta + 32\gamma].$$

Then (10) is derived.

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Appendix A. Proof of Lemma 1

π_1 is written corresponding to the following four cases.
Case 1

$$\pi_1(n, z, N - z) = \sum_{j=0}^{n-1} \frac{\binom{z-1}{j}\binom{N-z}{n-1-j}}{\binom{N-1}{n-1}} a_1(j+1), \text{ when } z - 1 \geqq n - 1, \text{ and } N - z \geqq n - 1.$$

Case 2

$$\pi_1(n, z, N - z) = \sum_{j=0}^{z-1} \frac{\binom{z-1}{j}\binom{N-z}{n-1-j}}{\binom{N-1}{n-1}} a_1(j+1), \text{ when } z - 1 < n - 1, \text{ and } N - z \geqq n - 1.$$

Case 3

$$\pi_1(n, z, N - z) = \sum_{j=n-1-N+z}^{n-1} \frac{\binom{z-1}{j}\binom{N-z}{n-1-j}}{\binom{N-1}{n-1}} a_1(j+1), \text{ when } z - 1 \geqq n - 1, \text{ and } N - z < n - 1.$$

Case 4

$$\pi_1(n, z, N - z) = \sum_{j=n-1-N+z}^{z-1} \frac{\binom{z-1}{j}\binom{N-z}{n-1-j}}{\binom{N-1}{n-1}} a_1(j+1), \text{ when } z - 1 < n - 1, \text{ and } N - z < n - 1. \quad (11)$$

Similarly, π_2 is written corresponding to the following four cases.
Case 1

$$\pi_2(n, z, N - z) = \sum_{j=0}^{n-1} \frac{\binom{z}{j}\binom{N-z-1}{n-1-j}}{\binom{N-1}{n-1}} a_2(j), \text{ when } z \geqq n - 1, \text{ and } N - z - 1 \geqq n - 1.$$

Case 2

$$\pi_2(n, z, N-z) = \sum_{j=0}^{z} \frac{\binom{z}{j}\binom{N-z-1}{n-1-j}}{\binom{N-1}{n-1}} a_2(j), \text{ when } z < n-1, \text{ and } N-z-1 \geqq n-1.$$

Case 3

$$\pi_2(n,z,N-z) = \sum_{j=n-N+z}^{n-1} \frac{\binom{z}{j}\binom{N-z-1}{n-1-j}}{\binom{N-1}{n-1}} a_2(j), \text{ when } z \geqq n-1, N-z-1 < n-1 \text{ and}$$

Case 4

$$\pi_2(n, z, N-z) = \sum_{j=n-N+z}^{z} \frac{\binom{z}{j}\binom{N-z-1}{n-1-j}}{\binom{N-1}{n-1}} a_2(j), \text{ when } z < n-1, \text{ and } N-z-1 < n-1. \tag{12}$$

Let's prove the lemma for Case 4 of $\pi_1(n, z, N-z)$ and Case 4 of $\pi_2(n, z, N-z)$. The proofs for other cases are similar.

1. When z increases from z to $z' = z+1$, the maximum payoff in $\pi_1(n, z, N-z)$ is increased from $a_1(z)$ to $a_1(z+1)$, and the minimum payoff is also increased from $a_1(n-N+z)$ to $a_1(n-N+z+1)$.

   The denominator of the coefficient of $a_1(j+1)$ in the summation in (11) is not affected by a change in z. The sum of the numerators of the coefficients of $a_1(j+1)$ over j is equal to the denominator, and it is constant. The numerator of the coefficient of $a_1(j+1)$ is rewritten as follows,

$$\binom{z-1}{j}\binom{N-z}{n-1-j} = \frac{\prod_{i=1}^{j}(z-i)}{\prod_{i=1}^{j} i} \times \frac{\prod_{i=1}^{n-1-j}(N-z-i+1)}{\prod_{i=1}^{n-1-j} i}. \tag{13}$$

   When z increases from z to $z' = z+1$, (13) is changed to

$$\binom{z'-1}{j}\binom{N-z'}{n-1-j} = \frac{\prod_{i=1}^{j}(z-i+1)}{\prod_{i=1}^{j} i} \times \frac{\prod_{i=1}^{n-1-j}(N-z-i)}{\prod_{i=1}^{n-1-j} i}. \tag{14}$$

   This is equal to

$$(13) \times \frac{z(N-z-n+1+j)}{(z-j)(N-z)}.$$

   By some calculations we find that (14) is larger than (13) if

$$j > \frac{z}{N}(n-1).$$

Thus the numerator of the coefficient of $a_1(j+1)$ with larger j is increased, and the numerator of the coefficient of $a_1(j+1)$ with smaller j is decreased by an increase in z. Note that $a_1(j)$ is increasing in j. Therefore $\pi_1(n, z, N-z)$ is increased by an increase in z.

2. When z increases from z to $z' = z+1$, the maximum payoff in $\pi_2(n, z, N-z)$ is decreased from $a_2(n-N+z)$ to $a_2(n-N+z+1)$, and the minimum payoff is also decreased from $a_2(z)$ to $a_2(z+1)$. Note that $a_2(k)$ is decreasing in k.

   The denominator of the coefficient of $a_2(j)$ in the summation in (12) is not affected by a change in z. The sum of the numerators of the coefficients of $a_2(j)$ over j is equal to the denominator, and it is constant. The numerator of the coefficient of $a_2(j)$ is rewritten as follows,

$$\binom{z}{j}\binom{N-z-1}{n-1-j} = \frac{\prod_{i=1}^{j}(z-i+1)}{\prod_{i=1}^{j} i} \times \frac{\prod_{i=1}^{n-1-j}(N-z-i)}{\prod_{i=1}^{n-1-j} i}. \tag{15}$$

When z increases from z to $z' = z+1$, (15) is changed to

$$\binom{z'}{j}\binom{N-z'-1}{n-1-j} = \frac{\prod_{i=1}^{j}(z-i+2)}{\prod_{i=1}^{j} i} \times \frac{\prod_{i=1}^{n-1-j}(N-z-i-1)}{\prod_{i=1}^{n-1-j} i}. \tag{16}$$

This is equal to

$$(15) \times \frac{(z+1)(N-z-n+j)}{(z-j+1)(N-z-1)}.$$

By some calculations we find that (16) is larger than (15) if

$$j > \frac{z+1}{N}(n-1).$$

Thus the numerator of the coefficient of $a_2(j)$ with larger j is increased, and the numerator of the coefficient of $a_2(j)$ with smaller j is decreased by an increase in z. Note that $a_2(j)$ is decreasing in j. Therefore $\pi_2(n, z, N-z)$ is decreased by an increase in z. □

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