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A topological approach to the Arrow impossibility theorem when individual preferences are weak orders

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Abstract

We will present a topological approach to the Arrow impossibility theorem of social choice theory that there exists no binary social choice rule (which we will call a social welfare function) which satisfies the conditions of transitivity, independence of irrelevant alternatives (IIA), Pareto principle and non-existence of dictator. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by Baryshnikov [Y. Baryshnikov, Unifying impossibility theorems: a topological approach, Advances in Applied Mathematics 14 (1993) 404–415]. But tools and techniques of algebraic topology which we will use are more elementary than those in Baryshnikov [Y. Baryshnikov, Unifying impossibility theorems: a topological approach, Advances in Applied Mathematics 14 (1993) 404–415]. Our main tools are homology groups of simplicial complexes. And we will consider the case where individual preferences are weak orders, that is, individuals may be indifferent about any pair of alternatives. This point is an extension of the analysis by Baryshnikov [Y. Baryshnikov, Unifying impossibility theorems: a topological approach, Advances in Applied Mathematics 14 (1993) 404–415].

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1. Introduction

Topological approaches to social choice problems have been initiated by Chichilnisky [5]. In her model a space of alternatives is a subset of a Euclidean space, and individual preferences over this set are represented by normalized gradient fields. Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. This approach has been further developed by Chichilnisky [4,6], Koshevoy [9], Lauwers [11], Weinberger [16], and so on. On the other hand, Baryshnikov [2,3] have presented a topological approach to Arrow's general possibility theorem, which is usually called the *Arrow impossibility theorem* [1], in a discrete framework of social choice.[1] But he used an advanced concept of algebraic topology, *nerve of a covering*. It is not dealt with in most elementary textbooks of algebraic topology, and is difficult of access for most economists. And he considered only the case where individual preferences are strict, that is, individuals are never indifferent about any pair of alternatives. In this paper we will attempt a more simple and elementary topological approach to the Arrow impossibility theorem under the assumption of the *free triple property*. Our main tool is the homology group of simplicial complexes. It is a basic concept of algebraic topology, and is dealt with in almost all elementary textbooks in this field. And we will consider the case where individual preferences are weak orders, that is, individuals may be indifferent about any pair of alternatives. This point is an extension of the analysis by Baryshnikov [2].

Mainly we will show the following results.

(1) Let Δ be an inclusion map from the set of individual preferences to the set of the social preference. Let i_i be an inclusion map from the set of the preference of individual i (a representative individual) to the set of the social preference, and F be a binary social choice rule (which we will call a social welfare function). Let $(F \circ \Delta)_*$ and $(F \circ i_i)_*$ be homomorphisms of homology groups induced by the composite functions of these inclusion maps and F. Then, we will obtain the following results:[2]

$$(F \circ \Delta)_* = \sum_{i=1}^{k} (F \circ i_i)_* \ (k \text{ is the number of individuals})$$

[1] About surveys and basic results of topological social choice theories, see [12,10].

[2] A homomorphism h is a mapping from a group A to another group B which satisfies $h(x + y) = h(x) + h(y)$ for $x \in A$, $y \in B$.

and

$$(F \circ \Delta)_* \neq 0.$$

(2) On the other hand, if social welfare functions satisfy the conditions of transitivity, Pareto principle, independence of irrelevant alternatives (IIA) and non-existence of dictator,[3] we can show

$$(F \circ i_i)_* = 0 \quad \text{for all } i,$$

(1) and (2) contradict. Therefore, there exists no binary social choice rule which satisfies transitivity, Pareto principle, IIA and non-existence of dictator.

In the next section we present our model and calculate the homology groups of simplicial complexes which represent individual preferences. In Section 3 we will prove the main results.

2. The model and simplicial complexes

There are n ($\geqslant 3$) alternatives and k ($\geqslant 2$) individuals. n and k are finite positive integers. Denote individual i's preference by p_i. A combination of individual preferences, which is called a preference profile, is denoted by $\mathbf{p}$, and the set of preference profiles is denoted by $\mathscr{P}^k$. The alternatives are represented by x_i, $i = 1, 2, \ldots, n$. Individual preferences over the alternatives are weak orders, that is, individuals strictly prefer one alternative to another, or are indifferent between them. We consider a social choice rule which determines a social preference corresponding to a preference profile. It is called a *social welfare function* and is denoted by $F(\mathbf{p})$. We assume the free triple property, that is, for each combination of three alternatives individual preferences are never restricted.

Social welfare functions must satisfy transitivity, Pareto principle and independence of irrelevant alternatives (IIA). The meanings of the latter two conditions are as follows:

Pareto principle. When all individuals prefer an alternative x_i to another alternative x_j, the society must prefer x_i to x_j.

Independence of irrelevant alternatives (*IIA*). The social preference about every pair of two alternatives x_i and x_j is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x_i and x_j.

[3] A dictator is an individual whose strict preference always coincide with the social preference.

The Arrow impossibility theorem states that there exists no binary social choice rule which satisfies the conditions of transitivity, IIA, Pareto principle and non-existence of dictator. A dictator is an individual whose strict preference always coincide with the social preference.

Hereafter we will consider a set of alternatives x_1, x_2 and x_3. From the set of individual preferences about x_1, x_2 and x_3 we construct a simplicial complex by the following procedures:

(1) A preference of an individual such that he prefers x_1 to x_2 is denoted by $(1,2)$, a preference such that he prefers x_2 to x_1 by $(2,1)$, a preference such that he is indifferent between x_1 and x_2 by $\overline{(1,2)}$, and similarly for other pairs of alternatives. Define vertices of the simplicial complex corresponding to (i,j) and $\overline{(i,j)}$.

(2) A line segment between the vertices (i,j) and (k,l) is included in the simplicial complex if and only if the preference represented by (i,j) and the preference represented by (k,l) are consistent, that is, they satisfy transitivity. For example, the line segment between $(1,2)$ and $(2,3)$ is included, but the line segment between $(1,2)$ and $(2,1)$ is not included in the simplicial complex. The line segment between $\overline{(1,2)}$ and $(2,3)$ is included, but the line segment between $(1,2)$ and $\overline{(1,2)}$ is not included in the simplicial complex.

(3) A triangle (circumference plus interior) made by three vertices (i,j), (k,l) and (m,n) is included in the simplicial complex if and only if the preferences represented by (i,j), (k,l) and (m,n) satisfy transitivity. For example, since the preferences represented by $(1,2)$, $(2,3)$ and $(1,3)$ satisfy transitivity, a triangle made by these three vertices is included in the simplicial complex. But, since the preferences represented by $(1,2)$, $(2,3)$ and $(3,1)$ do not satisfy transitivity, a triangle made by these three vertices is not included in the simplicial complex. Similarly, a triangle which includes a vertex $\overline{(i,j)}$ is included in the simplicial complex if and only if the vertices of that triangle satisfy transitivity. Since the preferences represented by $\overline{(1,2)}$, $(2,3)$ and $(1,3)$ satisfy transitivity, a triangle made by these three vertices is included in the simplicial complex. But, since the preferences represented by $\overline{(1,2)}$, $(2,3)$ and $(3,1)$ do not satisfy transitivity, a triangle made by these three vertices is not included in the simplicial complex.

The simplicial complex constructed by these procedures is denoted by P.

In Fig. 1 the simplicial complex made by preferences which do not include indifference (strict preferences) is depicted. This is called C_1. It is *homotopic* to a circumference of a circle (a 1-dimensional sphere S^1). The simplicial complex made by preferences which may include indifference is constructed by adding the following simplicial complexes to C_1.

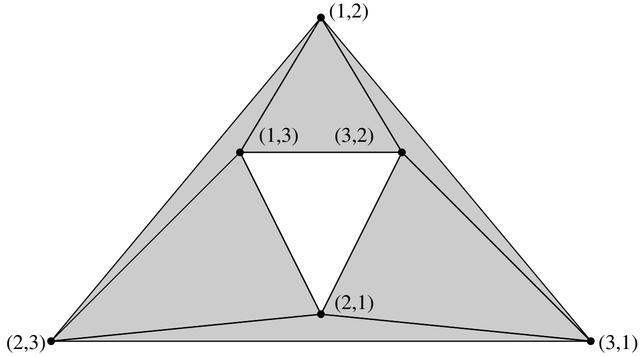

Fig. 1. The simplicial complex made by strict preferences (C_1).

The triangle made by $\overline{(1,2)}$, $(2,3)$, $(1,3)$ and its edges and vertices.
The triangle made by $\overline{(1,2)}$, $(3,2)$, $(3,1)$ and its edges and vertices.
The triangle made by $\overline{(1,3)}$, $(1,2)$, $(3,2)$ and its edges and vertices.
The triangle made by $\overline{(1,3)}$, $(2,1)$, $(2,3)$ and its edges and vertices.
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The triangle made by $\overline{(1,2)}$, $\overline{(2,3)}$, $\overline{(1,3)}$ and its edges and vertices.

The first two simplicial complexes are depicted in Fig. 2. This is called C_2. The latter five simplicial complexes are depicted in Fig. 3. This is called D. Let us denote $C = C_1 \cup C_2$.

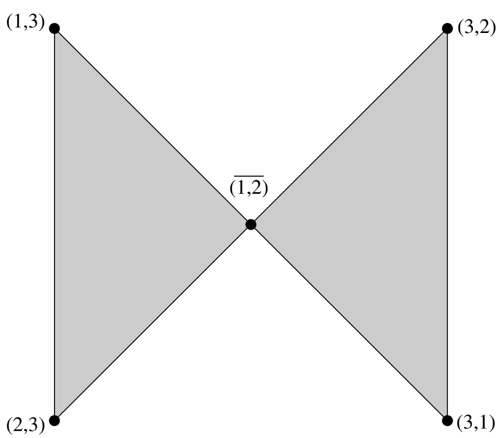

Fig. 2. C_2.

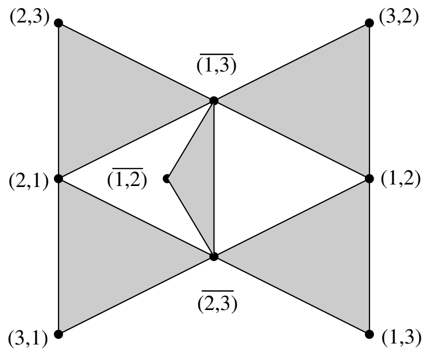

Fig. 3. *D*.

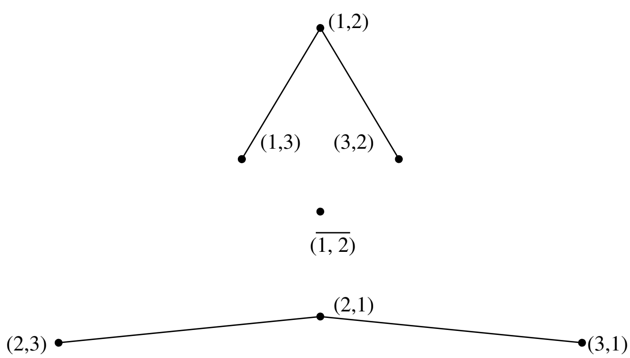

Fig. 4. E: (The intersection of *C* and *D*).

P is the union of *C* and *D*. The intersection of *C* and *D* is the graph depicted in Fig. 4. This is homotopic to isolated three points. It is denoted by *E*. Its 0-dimensional homology group is isomorphic to the group of three integers, and its 1-dimensional homology group is trivial, that is, $H_0(E) = \mathbb{Z}^3$ and $H_1(E) = 0$.

Now, we can show the following lemma.

Lemma 1. *The* 1-*dimensional homology group of P is isomorphic to the group of* 6 *integers*, *that is*, $H_1(P) \cong \mathbb{Z}^6$.

Proof. *P* contains the following 1-dimensional simplices:

$$\sigma_1 = \langle(1,2),(2,3)\rangle, \quad \sigma_2 = \langle(1,2),(3,2)\rangle, \quad \sigma_3 = \langle(1,2),(1,3)\rangle,$$
$$\sigma_4 = \langle(1,2),(3,1)\rangle, \quad \sigma_5 = \langle(2,1),(2,3)\rangle, \quad \sigma_6 = \langle(2,1),(3,2)\rangle,$$
$$\sigma_7 = \langle(2,1),(1,3)\rangle, \quad \sigma_8 = \langle(2,1),(3,1)\rangle, \quad \sigma_9 = \langle(2,3),(1,3)\rangle,$$

$$\sigma_{10} = \langle (2,3),(3,1) \rangle,\quad \sigma_{11} = \langle (3,2),(1,3) \rangle,\quad \sigma_{12} = \langle (3,2),(3,1) \rangle,$$
$$\sigma_{13} = \langle (\overline{1,2}),(2,3) \rangle,\quad \sigma_{14} = \langle (\overline{1,2}),(3,2) \rangle,\quad \sigma_{15} = \langle (\overline{1,2}),(1,3) \rangle,$$
$$\sigma_{16} = \langle (\overline{1,2}),(3,1) \rangle,\quad \sigma_{17} = \langle (\overline{2,3}),(1,2) \rangle,\quad \sigma_{18} = \langle (\overline{2,3}),(2,1) \rangle,$$
$$\sigma_{19} = \langle (\overline{2,3}),(1,3) \rangle,\quad \sigma_{20} = \langle (\overline{2,3}),(3,1) \rangle,\quad \sigma_{21} = \langle (\overline{1,3}),(1,2) \rangle,$$
$$\sigma_{22} = \langle (\overline{1,3}),(2,1) \rangle,\quad \sigma_{23} = \langle (\overline{1,3}),(2,3) \rangle,\quad \sigma_{24} = \langle (\overline{1,3}),(3,2) \rangle,$$
$$\sigma_{25} = \langle (\overline{1,2}),(\overline{2,3}) \rangle,\quad \sigma_{26} = \langle (\overline{1,2}),(\overline{1,3}) \rangle,\quad \sigma_{27} = \langle (\overline{2,3}),(\overline{1,3}) \rangle.$$

An element of the 1-dimensional chain group of P is written as follows:

$$c_1(P) = \sum_{i=1}^{27} a_i \sigma_i, \tag{1}$$

$a_1, a_2, \ldots, a_{27}$ are integers.

From this we obtain

$$\begin{aligned}
\partial c_1(P) = {} & (-a_1 - a_2 - a_3 - a_4 + a_{17} + a_{21})\langle (1,2) \rangle \\
& + (-a_5 - a_6 - a_7 - a_8 + a_{18} + a_{22})\langle (2,1) \rangle \\
& + (a_1 + a_5 - a_9 - a_{10} + a_{13} + a_{23})\langle (2,3) \rangle \\
& + (a_2 + a_6 - a_{11} - a_{12} + a_{14} + a_{24})\langle (3,2) \rangle \\
& + (a_3 + a_7 + a_9 + a_{11} + a_{15} + a_{19})\langle (1,3) \rangle \\
& + (a_4 + a_8 + a_{10} + a_{12} + a_{16} + a_{20})\langle (3,1) \rangle \\
& + (-a_{13} - a_{14} - a_{15} - a_{16} - a_{25} - a_{26})\langle (\overline{1,2}) \rangle \\
& + (-a_{17} - a_{18} - a_{19} - a_{20} + a_{25} - a_{27})\langle (\overline{2,3}) \rangle \\
& + (-a_{21} - a_{22} - a_{23} - a_{24} + a_{26} + a_{27})\langle (\overline{1,3}) \rangle.
\end{aligned}$$

The conditions for an element of the 1-dimensional chain group of P, $c_1(P)$, to be a cycle is $\partial c_1(P) = 0$. For this condition to hold all coefficients of $\partial c_1(P)$ must be zero, and we obtain the following equations:

$$\begin{aligned}
& -a_1 - a_2 - a_3 - a_4 + a_{17} + a_{21} = 0, \\
& -a_5 - a_6 - a_7 - a_8 + a_{18} + a_{22} = 0, \\
& a_1 + a_5 - a_9 - a_{10} + a_{13} + a_{23} = 0, \\
& a_2 + a_6 - a_{11} - a_{12} + a_{14} + a_{24} = 0, \\
& a_3 + a_7 + a_9 + a_{11} + a_{15} + a_{19} = 0, \\
& a_4 + a_8 + a_{10} + a_{12} + a_{16} + a_{20} = 0, \\
& -a_{13} - a_{14} - a_{15} - a_{16} - a_{25} - a_{26} = 0, \\
& -a_{17} - a_{18} - a_{19} - a_{20} + a_{25} - a_{27} = 0, \\
& -a_{21} - a_{22} - a_{23} - a_{24} + a_{26} + a_{27} = 0.
\end{aligned}$$

Summing up the first eight equations side by side we get the last equation. Therefore, only eight equations are independent, and we can freely choose the values of 19 variables among $a_1, a_2, \ldots, a_{27}$. Thus, the 1-dimensional cycle group of $P, Z_1(P)$, is isomorphic to the group of 19 integers, that is, $Z_1(P) \cong \mathbb{Z}^{19}$. P contains the following 2-dimensional simplices:

$$\tau_1 = \langle (1,2),(2,3),(1,3)\rangle, \quad \tau_2 = \langle (1,2),(3,2),(3,1)\rangle,$$
$$\tau_3 = \langle (1,2),(3,2),(1,3)\rangle, \quad \tau_4 = \langle (2,1),(2,3),(1,3)\rangle,$$
$$\tau_5 = \langle (2,1),(3,2),(3,1)\rangle, \quad \tau_6 = \langle (2,1),(2,3),(3,1)\rangle,$$
$$\tau_7 = \langle (\overline{1,2}),(2,3),(1,3)\rangle, \quad \tau_8 = \langle (\overline{1,2}),(3,2),(3,1)\rangle,$$
$$\tau_9 = \langle (\overline{2,3}),(1,2),(1,3)\rangle, \quad \tau_{10} = \langle (\overline{2,3}),(2,1),(3,1)\rangle,$$
$$\tau_{11} = \langle (\overline{1,3}),(1,2),(3,2)\rangle, \quad \tau_{12} = \langle (\overline{1,3}),(2,1),(2,3)\rangle,$$
$$\tau_{13} = \langle (\overline{1,2}),(\overline{2,3}),(\overline{1,3})\rangle.$$

An element of the 2-dimensional chain group of P is written as follows:

$$c_2(P) = \sum_{i=1}^{13} b_i \tau_i,$$

$b_1, b_2, \ldots, b_{13}$ are integers. The image of the boundary homomorphism of the 2-dimensional chain group of P is

$$\begin{aligned} \partial c_2(P) &= \sum_{i=1}^{13} b_i \partial \tau_i \\ &= b_1\sigma_1 + (b_2 + b_3 + b_{11})\sigma_2 + (-b_1 - b_3 + b_9)\sigma_3 - b_2\sigma_4 \\ &\quad + (b_4 + b_6 + b_{12})\sigma_5 + b_5\sigma_6 - b_4\sigma_7 + (-b_5 - b_6 + b_{10})\sigma_8 \\ &\quad + (b_1 + b_4 + b_7)\sigma_9 + b_6\sigma_{10} + b_3\sigma_{11} + (b_2 + b_5 + b_8)\sigma_{12} + b_7\sigma_{13} \\ &\quad + b_8\sigma_{14} - b_7\sigma_{15} - b_8\sigma_{16} + b_9\sigma_{17} + b_{10}\sigma_{18} - b_9\sigma_{19} - b_{10}\sigma_{20} \\ &\quad + b_{11}\sigma_{21} + b_{12}\sigma_{22} - b_{12}\sigma_{23} - b_{11}\sigma_{24} + b_{13}\sigma_{25} - b_{13}\sigma_{26} - b_{13}\sigma_{27}. \end{aligned} \quad (2)$$

The values of the coefficients of $\sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_6, \sigma_8, \sigma_9, \sigma_{12}, \sigma_{17}, \sigma_{18}, \sigma_{21}, \sigma_{22}, \sigma_{25}$ are determined by $b_1, b_2, \ldots, b_{13}$, and then the values of other σ's are also determined. Thus, the 1-dimensional boundary cycle group of P, $B_1(P)$, is isomorphic to the group of 13 integers, that is, $B_1(P) \cong \mathbb{Z}^{13}$. Therefore, the 1-dimensional homology group of P is isomorphic to the group of six integers, that is, we obtain $H_1(P) = Z_1(P)/B_1(P) \cong \mathbb{Z}^6$. $\square$

Next we consider the simplicial complex, P^k, made by the set of preference profiles of individuals, $\mathscr{P}^k$, about x_1, x_2 and x_3. We can show the following result.

Lemma 2. *The* 1*-dimensional homology group of* P^k *is isomorphic to the group of* 6k *integers, that is,* $H_1(P^k) \cong \mathbb{Z}^{6k}$.

Proof. As a preliminary result, we show $H_1(P \times C) \cong \mathbb{Z}^8$. Using C_1^1, C_1^2, C_2^1 and C_2^2 depicted in Figs. 5 and 6,[4] C is represented as $C = C^1 \cup C^2, C^1 = C_1^1 \cup C_2^1, C^2 = C_1^2 \cup C_2^2$. C^1 and C^2 are homotopic to one point, and the intersection of C^1 and C^2 consists of two segments and one point, which is denoted by G. G is homotopic to three isolated points, and we have $H_1(G) = 0$ and $H_0(G) \cong \mathbb{Z}^3$. From these arguments we obtain the following Mayer–Vietoris exact sequences,[5]

$$H_1(P \times G)(\cong (\mathbb{Z}^6)^3) \xrightarrow{k_1} H_1(P \times C^1) \oplus H_1(P \times C^2)(\cong \mathbb{Z}^6 \oplus \mathbb{Z}^6) \xrightarrow{w_1} H_1(P \times C)$$
$$\xrightarrow{\alpha_1} H_0(P \times G)(\cong \mathbb{Z}^3) \xrightarrow{k_0} H_0(P \times C^1) \oplus H_0(P \times C^2)(\cong \mathbb{Z} \oplus \mathbb{Z})$$
$$\xrightarrow{w_0} H_0(P \times C)(\cong \mathbb{Z}) \to 0.$$

Since w_0 is a surjection (onto mapping),[6] we have $\mathrm{Image}\, w_0 \cong \mathbb{Z}$. By the homomorphism theorem we obtain $H_0(P \times C^1) \oplus H_0(P \times C^2)/\mathrm{Ker}\, w_0 \cong \mathbb{Z}$, and then $\mathrm{Ker}\, w_0 \cong \mathbb{Z}$ is derived. Thus, from the condition of exact sequences we have $\mathrm{Image}\, k_0 \cong \mathrm{Ker}\, w_0 \cong \mathbb{Z}$. Again by the homomorphism theorem we obtain $H_0(P \times G)/\mathrm{Ker}\, k_0 \cong \mathrm{Image}\, k_0 \cong \mathbb{Z}$, and we get $\mathrm{Ker}\, k_0 \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus, we have $\mathrm{Image}\, \alpha_1 \cong \mathrm{Ker}\, k_0 \cong \mathbb{Z} \oplus \mathbb{Z}$, and by the homomorphism theorem $H_1(P \times C)/\mathrm{Ker}\, \alpha_1 \cong \mathbb{Z} \oplus \mathbb{Z}$ is derived. From the condition of exact sequences we have $\mathrm{Ker}\ \alpha_1 \cong \mathrm{Image}\, w_1$, and by the homomorphism theorem, $H_1(P \times C^1) \oplus H_1(P \times C^2)/\mathrm{Ker}\, w_1 \cong \mathrm{Image}\, w_1$ is derived. From the condition of exact sequences we obtain $\mathrm{Ker}\, w_1 \cong \mathrm{Image}\, k_1$. Now let us consider $\mathrm{Image}\, k_1$.

> Let x, y, z be the vertices of three connected components of G. Let $h \in H_1(P)$, then $h \times x \in H_1(P \times x)$, $h \times y \in H_1(P \times y)$ and $h \times z \in H_1(P \times z)$ belong to the different homology classes. Since C^1 is connected, there exists a sequence of 1-dimensional simplices connected x and y, and a sequence of 1-dimensional simplices connected x and z. Thus, they belong to the same homology class in $H_1(P \times C^1)$. We can show a similar result for $H_1(P \times C^2)$. Therefore we obtain $\mathrm{Image}\, k_1 \cong \mathbb{Z}^6$.

From $\mathrm{Ker}\, w_1 \cong \mathrm{Image}\, k_1$ we have $\mathrm{Ker}\, w_1 \cong \mathbb{Z}^6$, and from $H_1(P \times C^1) \oplus H_1(P \times C^2)/\mathrm{Ker}\, w_1 \cong \mathrm{Image}\, w_1$ we have $\mathrm{Image}\, w_1 \cong \mathbb{Z}^6$. Thus, $\mathrm{Ker}\, \alpha_1 \cong \mathbb{Z}^6$ is derived. Therefore, we obtain $H_1(P \times C) \cong \mathbb{Z}^8$. By similar procedures we can show $H_1(P \times D) \cong \mathbb{Z}^8$.

Using this result we will show $H_1(P^2) \cong \mathbb{Z}^{12}$. Since $P^2 = P \times (C \cup D) = (P \times C) \cup (P \times D)$, and $(P \times C) \cap (P \times D) = P \times E$ we obtain the following Mayer–Vietoris exact sequences:

[4] C_1^1 and C_1^2 are depicted in Fig. 5, and C_2^1 and C_2^2 are depicted in Fig. 6.

[5] About homology groups, the homomorphism theorem and the Mayer–Vietoris exact sequences we referred to [15,8].

[6] This is derived from the condition of exact sequences.

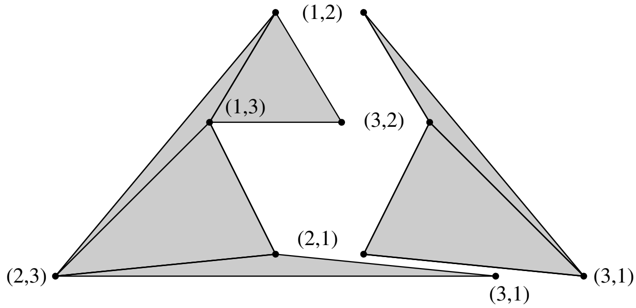

Fig. 5. C_1^1 and C_1^2.

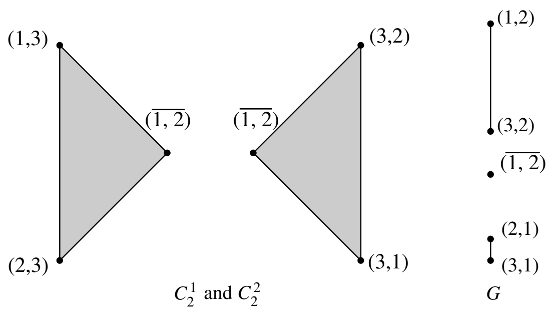

Fig. 6. C_2^1, C_2^2 and G.

$$H_1(P\times E)(\cong(\mathbb{Z}^6)^3)\xrightarrow{k_1}H_1(P\times C)\oplus H_1(P\times D)(\cong\mathbb{Z}^8\oplus\mathbb{Z}^8)\xrightarrow{w_1}H_1(P^2)$$
$$\xrightarrow{\alpha_1}H_0(P\times E)(\cong\mathbb{Z}^3)\xrightarrow{k_0}H_0(P\times C)\oplus H_0(P\times D)(\cong\mathbb{Z}\oplus\mathbb{Z})$$
$$\xrightarrow{w_0}H_0(P^2)(\cong\mathbb{Z})\to 0.$$

Since w_0 is a surjection, we have $\text{Image}\,w_0\cong\mathbb{Z}$. By the homomorphism theorem we obtain $H_0(P\times C)\oplus H_0(P\times D)/\text{Ker}\,w_0\cong\mathbb{Z}$, and $\text{Ker}\,w_0\cong\mathbb{Z}$ is derived. Thus, from the condition of exact sequences we have $\text{Image}\,k_0\cong\text{Ker}\,w_0\cong\mathbb{Z}$. Again by the homomorphism theorem we obtain $H_0(P\times E)/\text{Ker}\,k_0\cong\text{Image}\,k_0\cong\mathbb{Z}$, and $\text{Ker}\,k_0\cong\mathbb{Z}\oplus\mathbb{Z}$ is derived. Thus, we have $\text{Image}\,\alpha_1\cong\text{Ker}\,k_0\cong\mathbb{Z}\oplus\mathbb{Z}$, and by the homomorphism theorem $H_1(P^2)/\text{Ker}\,\alpha_1\cong\mathbb{Z}\oplus\mathbb{Z}$ is derived. Again from the condition of exact sequences we obtain $\text{Ker}\,\alpha_1\cong\text{Image}\,w_1$, and by the homomorphism theorem we obtain $H_1(P\times C)\oplus H_1(P\times D)/\text{Ker}\,w_1\cong\text{Image}\,w_1$. Further, from the condition of exact sequences $\text{Ker}\,w_1\cong\text{Image}\,k_1$ is derived. Now consider $\text{Image}\,k_1$.

Let x, y, z be the vertices of the connected components of E. Let $h \in H_1(P)$, then $h \times x \in H_1(P \times x)$, $h \times y \in H_1(P \times y)$ and $h \times z \in H_1(P \times z)$ belong to different homology classes. But, since C is connected, there exists a sequence of 1-dimensional simplices connecting x and y, and a sequence of 1-dimensional simplices connecting x and z. Thus, they belong to the same homology class in $H_1(P \times C)$. Similar for $H_1(P \times D)$. Therefore, we obtain $\text{Image}\, k_1 \cong \mathbb{Z}^6$.

From $\text{Ker}\, w_1 \cong \text{Image}\, k_1$ we have $\text{Ker}\, w_1 \cong \mathbb{Z}^6$. And from $H_1(P \times C) \oplus H_1(P \times D)/\text{Ker}\, w_1 \cong \text{Image}\, w_1$ we obtain $\text{Image}\, w_1 \cong \mathbb{Z}^{10}$. Thus, $\text{Ker}\, \alpha_1 \cong \mathbb{Z}^{10}$ is derived. Therefore, we get $H_1(P^2) \cong \mathbb{Z}^{12}$.

Inductively we can show $H_1(P^k) \cong \mathbb{Z}^{6k}$. □

The social preference is also represented by P. The social preference about x_i and x_j is (i,j) or (j,i) or $\overline{(i,j)}$. By the condition of IIA, individual preferences about alternatives other than x_i and x_j do not affect the social preference about them. Thus, the social welfare function F is a function from the vertices of P^k to the vertices of P. A set of points in P^k spans a simplex if and only if individual preferences represented by these points are consistent, that is, they satisfy transitivity, and then the social preference derived from the profile represented by these points also satisfies transitivity. Therefore, if a set of points in P^k spans a simplex, the set of points in P which represent the social preference corresponding to these points in P^k also spans a simplex in P, and hence the social welfare function is a *simplicial map*. It is naturally extended from the vertices of P^k to all points in P^k. Each point in P^k is represented as a convex combination of the vertices of P^k. This function is also denoted by F. When P represents the social preference, we denote it by P_s. Then, F is defined as a function from P^k to P_s.

We define an inclusion map from P to P^k, $\Delta : P \to P^k : p \to (p, p, \ldots, p)$, and an inclusion map which is derived by fixing preferences of individuals other than individual l to $\mathbf{p}_{-l}, i_l : P \to P^k : p \to (\mathbf{p}_{-l}, p)$. The homomorphisms of 1-dimensional homology groups induced by these inclusion maps are

$$\Delta_* : \mathbb{Z}^6 \to \mathbb{Z}^{6k} : h \to (h, h, \ldots, h), \quad h \in \mathbb{Z}^6,$$
$$i_{l*} : \mathbb{Z}^6 \to \mathbb{Z}^{6k} : \ h \to (0, \ldots, h, \ldots, 0)$$
$$\text{(only the } l\text{-th component is } h \text{ and others are zero, } h \in \mathbb{Z}^6).$$

From these definitions about Δ_* and i_{l*} we obtain the following relation:

$$\Delta_* = i_{1*} + i_{2*} + \cdots + i_{n*}. \tag{3}$$

And the homomorphism of homology groups induced by F is represented as follows:

$$F_* : \mathbb{Z}^{6k} \to \mathbb{Z}^6 : \mathbf{h} = (h_1, h_2, \ldots, h_n) \to h, \ h \in \mathbb{Z}^6.$$

The composite function of i_l and the social welfare function F is $F \circ i_l : P \to P_s$, and its induced homomorphism satisfies $(F \circ i_l)_* = F_* \circ i_{l*}$. The composite function of Δ and F is $F \circ \Delta : P \to P_s$, and its induced homomorphism satisfies $(F \circ \Delta)_* = F_* \circ \Delta_*$. From (3) we have

$$(F \circ \Delta)_* = (F \circ i_1)_* + (F \circ i_2)_* + \cdots + (F \circ i_n)_*,$$

$F \circ i_l$ when a preference profile of individuals other than individual l is $\mathbf{p}_{-l}$ and $F \circ i_l$ when a preference profile of individuals other than individual l is $\mathbf{p}'_{-l}$ are homotopic. Thus, the induced homomorphism $(F \circ i_l)_*$ of $F \circ i_l$ does not depend on the preferences of individuals other than l.

Note. Let $F \circ i_l(\mathbf{p}_{-l}, p_l)$ be the composite function of i_l and F when the preference profile of individuals other than l is $\mathbf{p}_{-l}$, and $F \circ i_l(\mathbf{p}'_{-l}, p_l)$ be the composite function of i_l and F when the preference profile of individuals other than l is $\mathbf{p}'_{-l}$. The component for one individual (denoted by k) of $\mathbf{p}_{-l}$ and that of $\mathbf{p}'_{-l}$ are denoted by p_k and p'_k. His preferences for the pair of alternatives x_i and x_j are denoted by $p_k(i,j)$ and $p'_k(i,j)$. Each of them corresponds to a point (i,j) or (j,i) or $(\overline{i,j})$ in P. Let (m,n) be a point in P such that $p_k(i,j)$ and $p'_k(i,j)$ are different from (m,n), (n,m) and $(\overline{m,n})$. Then, there exists a 1-dimensional simplex (a line segment) between $p_k(i,j)$ and (m,n), and a 1-dimensional simplex between $p'_k(i,j)$ and (m,n). Let

$$p''_k(i,j) = (1-2t)p_k(i,j) + 2t(m,n), \quad \text{if } 0 \leqslant t < \frac{1}{2},$$

$$p''_k(i,j) = (2t-1)p'_k(i,j) + (2-2t)(m,n), \quad \text{if } \frac{1}{2} \leqslant t \leqslant 1.$$

Then, $p''_k(i,j)$ is a point in P. Let us consider such $p''_k(i,j)$'s for all pairs of alternatives (x_i, x_j), and we denote a set of all $p''_k(i,j)$'s by p''_k. Similarly, p''_k's for all individuals other than k are also defined. Let $\mathbf{p}''_{-l}$ be a combination of p''_k's for all individuals other than l, and define

$$H(p,t) = F(\mathbf{p}''_{-l}, p_l).$$

Then, this is a homotopy between $F \circ i_l(\mathbf{p}_{-l}, p_l)$ and $F \circ i_l(\mathbf{p}'_{-l}, p_l)$.

Let $z = \langle(1,2),(2,3)\rangle + \langle(2,3),(3,1)\rangle - \langle(1,2),(3,1)\rangle$ be a cycle of P. By Pareto principle z corresponds to the same cycle in P_s by $(F \circ \Delta)_*$. Since it is not a boundary cycle, we have $(F \circ \Delta)_* \neq 0$.

Note. z is obtained by substituting $a_1 = 1$, $a_4 = -1$, $a_{10} = 1$ and 0 into all other coefficients of an element of the chain group of P expressed in (1). For this z to be a boundary of some 2-dimensional simplex we must have $b_1 = b_2 = b_6 = 1$ and $b_i = 0$ for all other coefficients of $\partial c_2(P)$ in (2). But then, $b_5, b_4, b_3, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}$ must be 0, and the coefficient of σ_2 is 1. Thus, z is not a boundary cycle.

For a pair of alternatives x_i and x_j, a preference profile, at which all individuals prefer x_i to x_j , is denoted by $(i,j)^{(+,+,...,+)}$; a preference profile, at which they prefer x_j to x_i, is denoted by $(i,j)^{(-,-,...,-)}$. Similarly a preference profile, at which all individuals other than l prefer x_i to x_j, is denoted by $(i,j)_{-l}^{(+,+,...,+)}$; a preference profile, at which they prefer x_j to x_i, is denoted by $(i,j)_{-l}^{(-,-,...,-)}$; a preference profile, at which they are indifferent between x_i and x_j, is denoted by $(i,j)_{-l}^{(0,0,...,0)}$. And a preference profile, at which the preferences of individuals other than l about x_i and x_j are not specified, is denoted by $(i,j)_{-l}^{(?,?,...,?)}$.

3. The main results

From preliminary analyses in the previous section we will show the following lemma.

Lemma 3

(1) *If individual l is the dictator, we have*

$$(F \circ i_l)_* \cong (F \circ \Delta)_*,$$

that is, $(F \circ i_l)_*$ and $(F \circ \Delta)_*$ are isomorphic.

(2) *If individual l is not a dictator, we have*

$$(F \circ i_l)_* = 0.$$

Proof. (1) Consider three alternatives x_1, x_2 and x_3 and a preference profile **p** over these alternatives such that the preferences of individuals other than l are represented by $(1,2)_{-l}^{(0,0,...,0)}$, $(2,3)_{-l}^{(0,0,...,0)}$ and $(1,3)_{-l}^{(0,0,...,0)}$, that is, they are indifferent about x_1, x_2 and x_3. If individual l is the dictator, correspondences from his preference to the social preference by $F \circ i_l$ are as follows:

$$\begin{aligned}&(1,2)_l \to (1,2),\ (2,1)_l \to (2,1),\\ &(2,3)_l \to (2,3),\ (3,2)_l \to (3,2),\\ &(1,3)_l \to (1,3),\ (3,1)_l \to (3,1),\end{aligned}$$

$(1,2)_l$ and $(2,1)_l$ denote the preference of individual l about x_1 and x_2. $(2,3)_l$, $(3,2)_l$ and so on are similar. These correspondences are completely identical to the correspondences by $F \circ \Delta$. Further, since we assume that individuals other than l are indifferent about x_1, x_2 and x_3, correspondences from the preferences of individual l, $\overline{(1,2)_l}, \overline{(2,3)_l}$ and $\overline{(1,3)_l}$, to the social preference by $F \circ i_l$ are also identical to the correspondences by $F \circ \Delta$. Therefore, the homomorphism of homology groups, $(F \circ \Delta)_*$ induced by $F \circ \Delta$, and the homomorphism of homology groups, $(F \circ i_l)_*$, which is induced by $F \circ i_l$, are identical (isomorphic), that is, $(F \circ i_l)_* \cong (F \circ \Delta)_*$.

(2) Consider three alternatives x_1, x_2 and x_3 and a preference profile **p** over these alternatives such that the preferences of individuals other than l are represented by $(1,2)_{-l}^{(+,+,\dots,+)}$, $(2,3)_{-l}^{(+,+,\dots,+)}$ and $(1,3)_{-l}^{(+,+,\dots,+)}$. If individual l is not a dictator, there exists a preference profile at which the social preference about some pair of alternatives does not coincide with the strict preference of individual l. Assume that when the preference of individual l is $(1,2)$, the social preference is $(2,1)$ or $(\overline{2,1})$. Then, we obtain the following correspondence from the preference profile to the social preference:

$$(1,2)_{-l}^{(?,?,\dots,?)} \times (1,2)_l \to (2,1) \text{ or } (\overline{2,1}).$$

By Pareto principle we have

$$(1,3)^{(+,+,\dots,+)} \to (1,3).$$

Then, from transitivity we obtain

$$(2,3)_{-l}^{(+,+,\dots,+)} \times (3,2)_l \to (2,3).$$

By Pareto principle we have

$$(1,2)^{(+,+,\dots,+)} \to (1,2).$$

From transitivity we obtain the following correspondence:

$$(1,3)_{-l}^{(+,+,\dots,+)} \times (3,1)_l \to (1,3).$$

Further, by Pareto principle we have

$$(2,3)^{(-,-,\dots,-)} \to (3,2).$$

From transitivity we get the following correspondence:

$$(1,2)_{-l}^{(+,+,\dots,+)} \times (2,1)_l \to (1,2).$$

From these results we find that at the preference profile **p**, where the preferences of individuals other than l are represented by $(1,2)_{-l}^{(+,+,\dots,+)}$, $(2,3)_{-l}^{(+,+,\dots,+)}$ and $(1,3)_{-l}^{(+,+,\dots,+)}$, correspondences from the preference of individual l to the social preference by $F \circ i_l$ are obtained as follows:

$$(1,2)_l \to (1,2),\ (2,1)_l \to (1,2),$$
$$(2,3)_l \to (2,3),\ (3,2)_l \to (2,3),$$
$$(1,3)_l \to (1,3),\ (3,1)_l \to (1,3).$$

From these correspondences with transitivity and IIA we find the following fact.

Sub-lemma 3.1.

When individual l is indifferent between x_1 and x_3, the society prefers x_1 to x_3, that is, we obtain the following correspondence:

$$(\overline{1,3})_l \to (1,3).$$

Proof. This is derived from two correspondences $(1,2)_l \to (1,2)$ and $(3,2)_l \to (2,3)$. □

Thus, the following four sets of correspondences are impossible because the correspondences in each set are not consistent with $(\overline{1,3})_l \to (1,3)$:

(i) $(\overline{1,2})_l \to (\overline{1,2}),\ (\overline{2,3})_l \to (\overline{2,3})$,
(ii) $(\overline{1,2})_l \to (\overline{1,2}),\ (\overline{2,3})_l \to (3,2)$,
(iii) $(\overline{1,2})_l \to (2,1),\ (\overline{2,3})_l \to (3,2)$,
(iv) $(\overline{1,2})_l \to (2,1),\ (\overline{2,3})_l \to (\overline{2,3})$.

And, we have the following five cases. They are consistent with the correspondence $(\overline{1,3})_l \to (1,3)$.

(i) Case (i): $(\overline{1,2})_l \to (\overline{1,2}),\ (\overline{2,3})_l \to (2,3)$,
(ii) Case (ii): $(\overline{1,2})_l \to (1,2),\ (\overline{2,3})_l \to (\overline{2,3})$,
(iii) Case (iii): $(\overline{1,2})_l \to (1,2),\ (\overline{2,3})_l \to (2,3)$,
(iv) Case (iv): $(\overline{1,2})_l \to (1,2),\ (\overline{2,3})_l \to (3,2)$,
(v) Case (v): $(\overline{1,2})_l \to (2,1),\ (\overline{2,3})_l \to (2,3)$.

We consider each case in detail.

(i) Case (i): $(\overline{1,2}) \to (\overline{1,2}),\ (\overline{2,3}) \to (2,3)$.

The vertices mapped by $F \circ i_l$ to the social preference from the preference of individual l span the following five simplices:

$$\langle(1,2),(2,3)\rangle,\ \langle(1,2),(1,3)\rangle,\ \langle(2,3),(1,3)\rangle,\ \langle(\overline{1,2}),(2,3)\rangle,\ \langle(\overline{1,2}),(1,3)\rangle.$$

Then, an element of the 1-dimensional chain group is written as

$$c_1 = a_1\langle(1,2),(2,3)\rangle + a_2\langle(1,2),(1,3)\rangle + a_3\langle(2,3),(1,3)\rangle \\ + a_4\langle(\overline{1,2}),(2,3)\rangle + a_5\langle(\overline{1,2}),(1,3)\rangle, \quad a_i \in \mathbb{Z}.$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2)\langle(1,2)\rangle + (a_1 - a_3 + a_4)\langle(2,3)\rangle + (a_2 + a_3 + a_5)\langle(1,3)\rangle \\ + (-a_4 - a_5)\langle(\overline{1,2})\rangle = 0.$$

From this

$$-a_1 - a_2 = 0, \quad a_1 - a_3 + a_4 = 0, \quad a_2 + a_3 + a_5 = 0, \quad -a_4 - a_5 = 0$$

are derived. Then, we obtain $a_2 = -a_1, a_5 = -a_4, a_3 = a_1 + a_4$. Therefore, an element of the 1-dimensional cycle group, Z_1, is written as follows:

$$z_1 = a_1\langle(1,2),(2,3)\rangle - a_1\langle(1,2),(1,3)\rangle + (a_1 + a_4)\langle(2,3),(1,3)\rangle + a_4\langle(\overline{1,2}),(2,3)\rangle - a_4\langle(\overline{1,2}),(1,3)\rangle.$$

On the other hand, the vertices span the following 2-dimensional simplices:

$$\langle(1,2),(2,3),(1,3)\rangle, \ \langle(\overline{1,2}),(2,3),(1,3)\rangle.$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1\langle(1,2),(2,3),(1,3)\rangle + b_2\langle(\overline{1,2}),(2,3),(1,3)\rangle, \quad b_i \in \mathbb{Z}.$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows:

$$\partial c_2 = b_1\langle(1,2),(2,3)\rangle - b_1\langle(1,2),(1,3)\rangle + (b_1 + b_2)\langle(2,3),(1,3)\rangle + b_2\langle(\overline{1,2}),(2,3)\rangle - b_2\langle(\overline{1,2}),(1,3)\rangle.$$

Then, we find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

(ii) Case (ii): $(\overline{1,2}) \to (1,2)$, $(\overline{2,3}) \to (\overline{2,3})$.
The vertices mapped by $F \circ i_l$ to the social preference from the preference of individual l span the following five simplices:

$$\langle(1,2),(2,3)\rangle, \ \langle(1,2),(1,3)\rangle, \ \langle(2,3),(1,3)\rangle, \ \langle(\overline{2,3}),(1,2)\rangle, \ \langle(\overline{2,3}),(1,3)\rangle.$$

Then, an element of the 1-dimensional chain group is written as

$$c_1 = a_1\langle(1,2),(2,3)\rangle + a_2\langle(1,2),(1,3)\rangle + a_3\langle(2,3),(1,3)\rangle + a_4\langle(\overline{2,3}),(1,2)\rangle + a_5\langle(\overline{2,3}),(1,3)\rangle.$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2 + a_4)\langle(1,2)\rangle + (a_1 - a_3)\langle(2,3)\rangle + (a_2 + a_3 + a_5)\langle(1,3)\rangle + (-a_4 - a_5)\langle(\overline{2,3})\rangle = 0.$$

From this

$$-a_1 - a_2 + a_4 = 0, \quad a_1 - a_3 = 0, \quad a_2 + a_3 + a_5 = 0, \quad -a_4 - a_5 = 0$$

are derived. Then, we obtain $a_3 = a_1$, $a_5 = -a_4$, $a_2 = a_4 - a_1$ Therefore, an element of the 1-dimensional cycle group, Z_1, is written as follows:

$$z_1 = a_1\langle(1,2),(2,3)\rangle + (a_4 - a_1)\langle(1,2),(1,3)\rangle + a_1\langle(2,3),(1,3)\rangle \\ + a_4\langle(\overline{2,3}),(1,2)\rangle - a_4\langle(\overline{2,3}),(1,3)\rangle.$$

On the other hand, the vertices span the following 2-dimensional simplices:

$$\langle(1,2),(2,3),(1,3)\rangle,\ \langle(\overline{2,3}),(1,2),(1,3)\rangle.$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1\langle(1,2),(2,3),(1,3)\rangle + b_2\langle(\overline{2,3}),(1,2),(1,3)\rangle.$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows:

$$\partial c_2 = b_1\langle(1,2),(2,3)\rangle + (b_2 - b_1)\langle(1,2),(1,3)\rangle + b_1\langle(2,3),(1,3)\rangle \\ + b_2\langle(\overline{2,3}),(1,2)\rangle - b_2\langle(\overline{2,3}),(1,3)\rangle.$$

We find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

(iii) Case (iii): $(\overline{1,2}) \to (1,2)$, $(\overline{2,3}) \to (2,3)$.
The vertices mapped by $F \circ i_l$ to the social preference from the preference of individual l span the following three simplices:

$$\langle(1,2),(2,3)\rangle,\ \langle(1,2),(1,3)\rangle,\ \langle(2,3),(1,3)\rangle.$$

Then, an element of the 1-dimensional chain group is written as

$$c_1 = a_1\langle(1,2),(2,3))\rangle + a_2\langle(1,2),(1,3)\rangle + a_3\langle(2,3),(1,3)\rangle.$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2)\langle(1,2)\rangle + (a_1 - a_3)\langle(2,3)\rangle + (a_2 + a_3)\langle(1,3)\rangle = 0.$$

From this

$$-a_1 - a_2 = 0, \quad a_1 - a_3 = 0, \quad a_2 + a_3 = 0$$

are derived, and we obtain $a_2 = -a_1$, $a_3 = a_1$. Therefore, an element of the 1-dimensional cycle group, Z_1, is written as follows:

$$z_1 = a_1\langle(1,2),(2,3)\rangle - a_1\langle(1,2),(1,3)\rangle + a_1\langle(2,3),(1,3)\rangle.$$

On the other hand, the vertices span the following 2-dimensional simplex.

$$\langle(1,2),(2,3),(1,3)\rangle.$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1\langle(1, 2), (2, 3), (1, 3)\rangle.$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows:

$$\partial c_2 = b_1\langle(1, 2), (2, 3)\rangle - b_1\langle(1, 2), (1, 3)\rangle + b_1\langle(2, 3), (1, 3)\rangle.$$

We find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

(iv) Case (iv): $(\overline{1, 2}) \to (1, 2)$, $(\overline{2, 3}) \to (3, 2)$.
The vertices mapped by $F \circ i_l$ to the social preference from the preference of individual l span the following five simplices:

$$\langle(1,2),(2,3)\rangle,\ \langle(1,2),(1,3)\rangle,\ \langle(2,3),(1,3)\rangle,\ \langle(3,2),(1,2)\rangle,\ \langle(3,2),(1,3)\rangle.$$

Then, an element of the 1-dimensional chain group is written as

$$c_1 = a_1\langle(1, 2), (2, 3)\rangle + a_2\langle(1, 2), (1, 3)\rangle + a_3\langle(2, 3), (1, 3)\rangle + a_4\langle(3, 2), (1, 2)\rangle + a_5\langle(3, 2), (1, 3)\rangle.$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2 + a_4)\langle(1, 2)\rangle + (a_1 - a_3)\langle(2, 3)\rangle + (a_2 + a_3 + a_5)\langle(1, 3)\rangle + (-a_4 - a_5)\langle(3, 2)\rangle = 0.$$

From this

$$-a_1 - a_2 + a_4 = 0, \quad a_1 - a_3 = 0, \quad a_2 + a_3 + a_5 = 0, \quad -a_4 - a_5 = 0$$

are derived, and we obtain $a_3 = a_1$, $a_5 = -a_4$, $a_2 = a_4 - a_1$. Therefore, an element of the 1-dimensional cycle group, Z_1, is written as follows:

$$z_1 = a_1\langle(1, 2), (2, 3)\rangle + (a_4 - a_1)\langle(1, 2), (1, 3)\rangle + a_1\langle(2, 3), (1, 3)\rangle + a_4\langle(3, 2), (1, 2)\rangle - a_4\langle(3, 2), (1, 3)\rangle.$$

On the other hand, the vertices span the following 2-dimensional simplices:

$$\langle(1, 2), (2, 3), (1, 3)\rangle,\ \langle(3, 2), (1, 2), (1, 3)\rangle.$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1\langle(1, 2), (2, 3), (1, 3)\rangle + b_2\langle(3, 2), (1, 2), (1, 3)\rangle.$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows:

$$\partial c_2 = b_1\langle(1,2),(2,3)\rangle + (b_2 - b_1)\langle(1,2),(1,3)\rangle + b_1\langle(2,3),(1,3)\rangle + b_2\langle(3,2),(1,2)\rangle - b_2\langle(3,2),(1,3)\rangle.$$

We find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

(v) Case (v): $\overline{(1,2)} \to (2,1)$, $\overline{(2,3)} \to (2,3)$.
The vertices mapped by $F \circ i_l$ to the social preference from the preference of individual l span the following five simplices:

$$\langle(1,2),(2,3)\rangle,\ \langle(1,2),(1,3)\rangle,\ \langle(2,3),(1,3)\rangle,\ \langle(2,1),(2,3)\rangle,\ \langle(2,1),(1,3)\rangle.$$

Then, an element of the 1-dimensional chain group is written as

$$c_1 = a_1\langle(1,2),(2,3)\rangle + a_2\langle(1,2),(1,3)\rangle + a_3\langle(2,3),(1,3)\rangle + a_4\langle(2,1),(2,3)\rangle + a_5\langle(2,1),(1,3)\rangle.$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2)\langle(1,2)\rangle + (a_1 - a_3 + a_4)\langle(2,3)\rangle + (a_2 + a_3 + a_5)\langle(1,3)\rangle + (-a_4 - a_5)\langle(2,1)\rangle = 0.$$

From this

$$-a_1 - a_2 = 0, \quad a_1 - a_3 + a_4 = 0, \quad a_2 + a_3 + a_5 = 0, \quad -a_4 - a_5 = 0$$

are derived, and we obtain $a_2 = -a_1$, $a_5 = -a_4$, $a_3 = a_1 + a_4$. Therefore, an element of the 1-dimensional cycle group is represented as follows:

$$z_1 = a_1\langle(1,2),(2,3)\rangle - a_1\langle(1,2),(1,3)\rangle + (a_1 + a_4)\langle(2,3),(1,3)\rangle + a_4\langle(2,1),(2,3)\rangle - a_4\langle(2,1),(1,3)\rangle.$$

On the other hand, the vertices span the following 2-dimensional simplices:

$$\langle(1,2),(2,3),(1,3)\rangle,\ \langle(2,1),(2,3),(1,3)\rangle.$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1\langle(1,2),(2,3),(1,3)\rangle + b_2\langle(2,1),(2,3),(1,3)\rangle.$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows:

$$\partial c_2 = b_1\langle(1,2),(2,3)\rangle - b_1\langle(1,2),(1,3)\rangle + (b_1 + b_2)\langle(2,3),(1,3)\rangle + b_2\langle(2,1),(2,3)\rangle - b_2\langle(2,1),(1,3)\rangle.$$

We find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

We have completely proved $(F \circ i_l)_* = 0$ in all cases. □

From these arguments and $(F \circ \Delta)_* \neq 0$ there exists the dictator about x_1, x_2 and x_3. Let individual l be the dictator. Interchanging x_3 with x_4 in the proof of this lemma, we can show that he is the dictator about x_1, x_2 and x_4. Similarly, we can show that he is the dictator about x_5, x_2 and x_4, he is the dictator about x_5, x_6 and x_4. After all he is the dictator about all alternatives, and hence we obtain

Theorem 1 (The Arrow impossibility theorem). *There exists the dictator for any social welfare function which satisfies transitivity, Pareto principle and IIA.*

4. Concluding remarks

We have shown the Arrow impossibility theorem when individual preferences are weak orders under the assumption of free-triple property using elementary concepts and techniques of algebraic topology, in particular, homology groups of simplicial complexes and homomorphisms of homology groups induced by simplicial maps.

Our approach may be applied to other problems of social choice theory such as Wilson's impossibility theorem [17], the Gibbard–Satterthwaite theorem [7,13], and Amartya Sen's liberal paradox [14].

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