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# On the equivalence of the Arrow impossibility theorem and the Brouwer fixed point theorem

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## Abstract

We will show that in the case where there are two individuals and three alternatives (or under the assumption of free-triple property) the Arrow impossibility theorem [K.J. Arrow, *Social Choice and Individual Values*, second ed., Yale University Press, 1963] for social welfare functions that there exists no social welfare function which satisfies transitivity, Pareto principle, independence of irrelevant alternatives, and has no dictator is equivalent to the Brouwer fixed point theorem on a 2-dimensional ball (circle). Our study is an application of ideas by Chichilnisky [*Economics Letters* 3 (1979) 347–351] to a discrete social choice problem, and also it is in line with the work by Baryshnikov [*Advances in Applied Mathematics* 14 (1993) 404–415]. But tools and techniques of algebraic topology which we will use are more elementary than those in Baryshnikov [*Advances in Applied Mathematics* 14 (1993) 404–415].

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*Keywords:* The Arrow impossibility theorem; Homology groups of simplicial complexes; The Brouwer fixed point theorem

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## 1. Introduction

Topological approaches to social choice problems have been initiated by Chichilnisky [5]. In her model a space of alternatives is a subset of Euclidean space, and individual preferences over this set are represented by normalized gradient fields. Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. This approach has been further developed by Chichilnisky [4,6], Koshevoy [8], Lauwers [10], and so on. In particular, by Chichilnisky [4] the equivalence of her impossibility result and the Brouwer fixed point theorem in the case where there are two individuals and the choice space is a subset of 2-dimensional Euclidean space has been shown. On the other hand, Baryshnikov [2,3] have presented a topological approach to Arrow's general possibility theorem, which is usually called the *Arrow impossibility theorem* [1], in a discrete framework of social choice.<sup>1</sup>

We will examine the relationship between the Arrow impossibility theorem for social welfare functions that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator,<sup>2</sup> and the Brouwer fixed point theorem on a 2-dimensional ball in the case of two individuals and three alternatives (or under the assumption of free-triple property).<sup>3</sup> Our study is an application of ideas by Chichilnisky [4] to a discrete social choice problem, and also it is in line with the work by Baryshnikov [2]. But tools and techniques of algebraic topology which we will use are more elementary than those used in Baryshnikov [2]. He used an advanced concept of algebraic topology, *nerve of a covering*. It is not contained in most elementary textbooks of algebraic topology, and is difficult of access for most economists. Our main tools are homology groups of simplicial complexes. Of course, the Brouwer fixed point theorem is a theorem about continuous functions. We will consider a method to obtain a continuous function from a discrete social choice rule. Mainly we will show the following results:

- (1) The Brouwer fixed point theorem is equivalent to the result that the restriction to an  $n - 1$ -dimensional sphere  $S^{n-1}$  of a continuous function from an  $n$ -dimensional ball  $D^n$  to  $S^{n-1}$  is homotopic to a constant mapping.
- (2) The restriction of a continuous function obtained from a social welfare function which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator to a subset of the set of profiles

<sup>1</sup> About surveys and basic results of topological social choice theories, see [11,9].

<sup>2</sup> Dictator is an individual whose (strict) preference always coincide with the social preference.

<sup>3</sup> Under the assumption of free-triple property, for each combination of three alternatives individual preferences are not restricted.

of individual preferences, which is homeomorphic to a 2-dimensional ball (or circle) and the subset is homeomorphic to a 1-dimensional sphere (or circumference), is not homotopic to a constant mapping. It implies that the non-existence of social welfare function which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator is equivalent to the Brouwer fixed point theorem on a 2-dimensional ball.

In the next section we present the model of this paper, and consider the homology groups of simplicial complexes which represent the set of individual preferences and the set of the social preference. In Section 3 we will show a result about the Brouwer fixed point theorem and homotopy of continuous functions. In Section 4 we will prove the main results.

## 2. The model

There are two individuals, A and B, and three alternatives of a social, economic or political problem,  $x_1$ ,  $x_2$  and  $x_3$  (or we assume free-triple property). Individual preferences about these alternatives are not restricted. We assume that individual preferences for these alternatives are linear, that is, their preferences are always strict, and they are never indifferent about any pair of alternatives. Individual preferences must be complete and transitive. A social choice rule which we will consider is a rule which determines a preference of the society about  $x_1$ ,  $x_2$  and  $x_3$  corresponding to a combination of preferences of two individuals. This social choice rule is called a *social welfare function*. We require that social welfare functions satisfy transitivity, Pareto principle and independence of irrelevant alternatives. The means of the latter two conditions are as follows:

*Pareto principle*: If all individuals prefer an alternative  $x_i$  to another alternative  $x_j$ , then the society must prefer  $x_i$  to  $x_j$ .

*Independence of irrelevant alternatives*: The social preference about any pair of two alternatives  $x_i$  and  $x_j$  is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about  $x_i$  and  $x_j$ .

The Arrow impossibility theorem states that there exists a dictator for any social welfare function which satisfies transitivity, Pareto principle and independence of irrelevant alternatives, or in other words there exists no social welfare function which satisfies these conditions and has no dictator. Dictator is an individual whose (strict) preference always coincide with the social preference.

From the set of individual preferences we draw a diagram by the following procedures:

- (1) When an individual prefers  $x_1$  to  $x_2$  to  $x_3$ , such a preference is denoted by (123), and corresponding to this preference we define a vertex  $v_1$ . Similarly, when an individual prefers  $x_1$  to  $x_3$  to  $x_2$ , such a preference is denoted by (132), and we define a vertex  $v_2$ . By similar procedures the following vertices are defined:

$$v_1 = (123), \quad v_2 = (132), \quad v_3 = (312), \quad v_4 = (321), \quad v_5 = (231), \\ v_6 = (213).$$

For example,  $v_6 = (213)$  denotes a preference of an individual such that he prefers  $x_2$  to  $x_1$  to  $x_3$ .

- (2) These six vertices are plotted on a line segment in this order, locate  $v_1$  at both end points, and connect the vertices.

Denote this diagram by  $R$ , and call  $v_1, v_2, \dots, v_6$  the vertices of  $R$ . It is depicted in Fig. 1.

Two  $v_1$ 's at both end points of  $R$  are not distinguished. The set of individual preferences is represented by  $R$ , and the set of combinations of the preferences of two individuals is represented by the product space  $R \times R$ . These combinations of individual preferences are called *preference profiles*.  $R \times R$  is depicted as a square in Fig. 2. The preference of individual B is represented from bottom up, not from left to right. Individual preferences are denoted by  $p_A = v_1, p_B = v_2$  and so on, and preference profiles are denoted by  $\mathbf{p} = (p_A, p_B) = (v_1, v_3)$ , and so on.

The social preference is represented by a circumference depicted in Fig. 3. We call this circumference  $S^1$ . The vertices of  $S^1$  are denoted by  $w_1, w_2, \dots, w_6$ . These vertices mean the following social preferences:<sup>4</sup>

- (1)  $w_1$ : The society prefers  $x_1$  to  $x_2, x_2$  to  $x_3$ .
- (2)  $w_2$ : The society prefers  $x_1$  to  $x_3, x_3$  to  $x_2$ .
- (3)  $w_3$ : The society prefers  $x_3$  to  $x_1, x_1$  to  $x_2$ .
- (4)  $w_4$ : The society prefers  $x_3$  to  $x_2, x_2$  to  $x_1$ .
- (5)  $w_5$ : The society prefers  $x_2$  to  $x_3, x_3$  to  $x_1$ .
- (6)  $w_6$ : The society prefers  $x_2$  to  $x_1, x_1$  to  $x_3$ .

The 1-dimensional homology group (with integer coefficients) of  $S^1$  is isomorphic to the group of integers  $\mathbb{Z}$ , that is, we have  $H_1(S^1) \cong \mathbb{Z}$ .

A social welfare function  $F$  is defined as a function from the vertices of  $R \times R$  to the vertices of  $S^1$ . Let us consider a method to obtain a continuous

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<sup>4</sup> From Lemma 1 of [2] we know that if individual preferences are strict orders, then the social preference is also a strict order under transitivity, Pareto principle and independence of irrelevant alternatives.

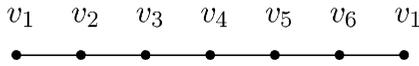


Fig. 1.  $R$ .

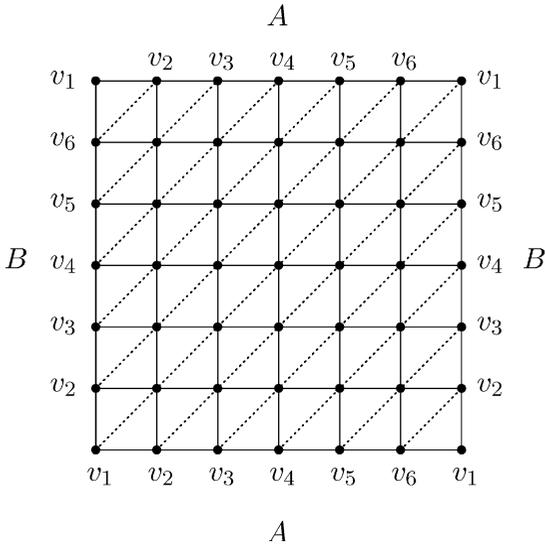


Fig. 2.  $R \times R$ .

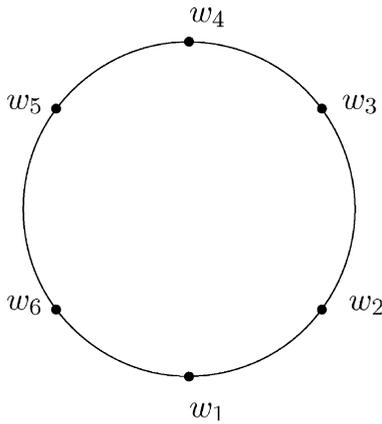


Fig. 3.  $S^1$ .

function from a social welfare function defined on the vertices of  $R \times R$ . For example, for points included in a small triangle which consists of  $(v_1, v_3)$ ,  $(v_2, v_3)$  and  $(v_2, v_4)$  we define

$$F(\alpha(v_1, v_3) + \beta(v_2, v_3) + \gamma(v_2, v_4)) = \alpha F(v_1, v_3) + \beta F(v_2, v_3) + \gamma F(v_2, v_4),$$

where  $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$  and  $0 \leq \gamma \leq 1, \alpha + \beta + \gamma = 1$ . Then, we can obtain a continuous function for the points in this triangle. By similar ways this continuous function is extended to the entire  $R \times R$ , and we obtain a continuous function for all points in  $R \times R$  from a discrete social welfare function on the vertices of  $R \times R$ . Denote this continuous function by  $F: R \times R \rightarrow S^1$ .

Let us see that this continuous function is well defined for the entire  $R \times R$ . By independence of irrelevant alternatives, for example, if  $F(v_1, v_3) = w_1$ , we must have  $F(v_2, v_3) = w_1$  or  $F(v_2, v_3) = w_2$ . As this example shows, preferences represented by adjacent two vertices of  $R \times R$  are identical about two pairs of alternatives. When the preference of one of two individuals change, the social preference does not change, or it changes to one of adjacent vertices. Therefore,  $F$  is a *simplicial mapping*. If the preferences of two individuals change, the social preference moves at most two vertices clockwise or counter-clockwise on  $S^1$ , and hence the social preference does not change to the antipodal point or across the antipodal point on  $S^1$ . Thus,  $\alpha F(v_1, v_3) + \beta F(v_2, v_3) + \gamma F(v_2, v_4)$  is well defined. Other cases are similar. Since  $F$  defined on the vertices of  $R \times R$  is a simplicial mapping, we can define the homomorphism of homology groups induced by  $F$ . It is denoted by  $F_*$ .

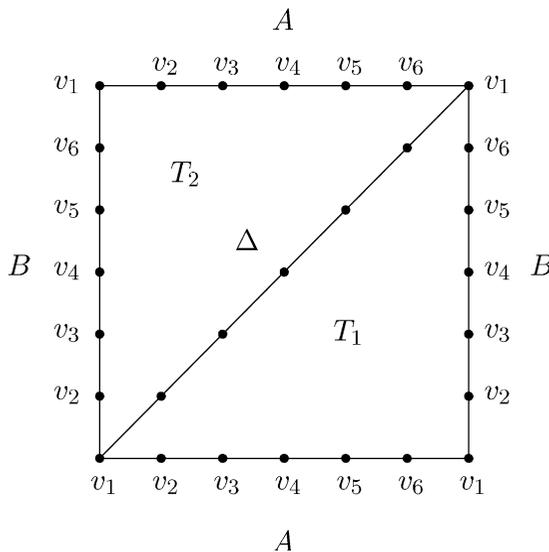


Fig. 4.  $\Delta \cup A \cup B$  and  $R \times R$ .

Now we consider the following set  $\Delta$  of vertices of  $R \times R$ :

$$\Delta = \{(v_1, v_1), (v_2, v_2), (v_3, v_3), (v_4, v_4), (v_5, v_5), (v_6, v_6), (v_1, v_1)\}.$$

The diagram obtained by connecting these vertices is also denoted by  $\Delta$ . It is homeomorphic to  $R$ . Preference profiles of two individuals when the preference of individual B is fixed at  $v_1$ , and preference profiles when the preference of individual A is fixed at  $v_1$  are denoted, respectively, by  $A = \{(p_A, p_B) | p_B = v_1\}$  and  $B = \{(p_A, p_B) | p_A = v_1\}$ . The diagrams obtained by connecting vertices of  $A$ , and similarly obtained from  $B$  are also denoted, respectively, by  $A$  and  $B$ . They are also homeomorphic to  $R$ . The union of these three sets  $\Delta \cup A \cup B$  is depicted as the boundary  $\partial T_1$  of the triangle  $T_1$  in Fig. 4.  $\Delta \cup A \cup B$  is homeomorphic to the circumference  $S^1$ . The vertices at four corners of the square depicted in Fig. 4 represent the same profile  $(v_1, v_1)$ . The value of  $F$  for them are equal. The 1-dimensional homology group of  $\Delta \cup A \cup B$  isomorphic to  $\mathbb{Z}$ , that is,  $H_1(\Delta \cup A \cup B) \cong \mathbb{Z}$ .

### 3. The Brouwer fixed point theorem

In this section we show the following theorem about homotopy and the degree of mapping of a continuous function on an  $n - 1$ -dimensional sphere.

*Note.* Let  $F$  be a function from  $n - 1$ -dimensional sphere  $S^{n-1}$  to itself, and  $F_*$  be the homomorphism of homology groups induced by  $F$ ,

$$F_* : H_{n-1}(S^{n-1}) \rightarrow H_{n-1}(S^{n-1}),$$

$H_{n-1}(S^{n-1})$  is the  $n - 1$ -dimensional homology group of  $S^{n-1}$ . Then, the degree of mapping of  $F$ , which is denoted by  $d_F$ , is defined as an integer which satisfies

$$F_*(h) = d_F h \quad \text{for } h \in H_{n-1}(S^{n-1}).$$

**Theorem 1.** *The following two results are equivalent:*

- (1) *If there exists a continuous function from an  $n$ -dimensional ball  $D^n$  to an  $n - 1$ -dimensional sphere  $S^{n-1}$  ( $n \geq 2$ ),  $F: D^n \rightarrow S^{n-1}$ , then the following function, which is obtained by restricting  $F$  to the boundary  $S^{n-1}$  of  $D^n$ ,*

$$F|_{S^{n-1}} : S^{n-1} \rightarrow S^{n-1}$$

*is homotopic to a constant mapping. Since the degree of mapping of a constant mapping is zero, the degree of mapping of  $F|_{S^{n-1}}$  is zero.*

- (2) *(The Brouwer fixed point theorem) Any continuous function from  $D^n$  to  $D^n$  ( $n \geq 2$ ),  $G: D^n \rightarrow D^n$ , has a fixed point.*

**Proof.** (1)  $\rightarrow$  (2)

Assume that  $G$  has no fixed point. Since we always have  $v \neq G(v)$  at any point  $v$  in  $D^n$ , there is a half line starting  $G(v)$  across  $v$ .<sup>5</sup> Let  $F(v)$  be the intersection point of this half line and the boundary of  $D^n$ , which is  $S^{n-1}$ . Then, we obtain the following continuous function from  $D^n$  to  $S^{n-1}$

$$F : D^n \rightarrow S^{n-1}.$$

In particular, we have  $F(v) = v$  for  $v \in S^{n-1}$ . Therefore,  $F|_{S^{n-1}}$  is an identity mapping. But, because an identity mapping on  $S^{n-1}$  is not homotopic to any constant mapping, it is a contradiction.

(2)  $\rightarrow$  (1)

We show that if there exists a continuous function  $F$  from  $D^n$  to  $S^{n-1}$ , (1) of this theorem is correct whether a continuous function  $G$  from  $D^n$  to  $D^n$  has a fixed point or not. Define  $f_t(v) = F[(1-t)v]$  ( $0 \leq t \leq 1$ ) for any point  $v$  of  $S^{n-1}$ . Then, we get a continuous function  $f_t : S^{n-1} \rightarrow S^{n-1}$ .  $(1-t)v$  is a point which divide  $t:1-t$  a line segment between  $v$  and the center of  $D^n$ , and it is transferred by  $F$  to a point on  $S^{n-1}$ . We have  $f_0 = F|_{S^{n-1}}$ , and  $f_1 = F(0)$  is a constant mapping whose image is a point  $F(0)$ . Since  $f_t$  is continuous with respect to  $t$ , it is a homotopy from  $F|_{S^{n-1}}$  to a constant mapping, and the degree of mapping of  $F|_{S^{n-1}}$  is zero.  $\square$

An implication of this theorem is as follows:

**Corollary 1.** *If there exists a function from  $D^n$  to  $S^{n-1}$ ,  $F: D^n \rightarrow S^{n-1}$ , and its restriction to  $S^{n-1}$ ,  $F|_{S^{n-1}} : S^{n-1} \rightarrow S^{n-1}$ , is not homotopic to a constant mapping,  $F$  cannot be continuous.*

In relation to a social welfare function on  $R \times R$ , if there exists a function  $F$  defined on the vertices of  $R \times R$ , we can obtain a continuous function on the entire  $R \times R$  from  $F$  by the way explained above. Then, there exists a continuous function defined on  $T_1$ . Since  $T_1$  is homeomorphic to  $D^2$  (2-dimensional ball), and  $A \cup A \cup B$  is homeomorphic to  $S^1$  (1-dimensional sphere), the restriction of  $F$  to  $A \cup A \cup B$ ,  $F|_{A \cup A \cup B}$ , must be homotopic to a constant mapping. If, when we require that transitivity, Pareto principle, independence of irrelevant alternatives and the non-existence of dictator are satisfied by a social welfare function defined on the vertices of  $R \times R$ , the restriction of this function to  $A \cup A \cup B$  is not homotopic to a constant mapping, then there does not exist such a social welfare function in the first place.

<sup>5</sup> If  $v$  is a fixed point,  $G(v)$  and  $v$  coincide, and hence there does not exist such a half line.

#### 4. The main results

From the preliminary analyses in the previous sections we can show the following lemma:

**Lemma 1.** *Suppose that there exists a social welfare function  $F: R \times R \rightarrow S^1$  which satisfies transitivity, Pareto principle and independence of irrelevant alternatives. If  $F$  has no dictator, then the degree of mapping of  $F|_{\Delta \cup A \cup B}$  is not zero, and hence it is not homotopic to a constant mapping.*

**Proof.** By Pareto principle the vertices of  $\Delta$  correspond to the vertices of  $S^1$  as follows:

$$\begin{aligned} (v_1, v_1) &\rightarrow w_1, & (v_2, v_2) &\rightarrow w_2, & (v_3, v_3) &\rightarrow w_3, \\ (v_4, v_4) &\rightarrow w_4, & (v_5, v_5) &\rightarrow w_5, & (v_6, v_6) &\rightarrow w_6. \end{aligned}$$

Next, also by Pareto principle,  $(v_2, v_1)$  corresponds to  $w_1$  or  $w_2$  in  $S^1$ . First, assume

$$(v_2, v_1) \rightarrow w_2. \tag{1}$$

(1) means that when individual A prefers  $x_3$  to  $x_2$  and individual B prefers  $x_2$  to  $x_3$ , then the society prefers  $x_3$  to  $x_2$ . By Pareto principle, independence of irrelevant alternatives, and transitivity we have

$$(v_4, v_6) \rightarrow w_4.$$

This means that when individual A prefers  $x_3$  to  $x_1$  and individual B prefers  $x_1$  to  $x_3$ , then the society prefers  $x_3$  to  $x_1$ . Similarly, we get

$$(v_5, v_1) \rightarrow w_5.$$

This means that when individual A prefers  $x_2$  to  $x_1$  and individual B prefers  $x_1$  to  $x_2$ , then the society prefers  $x_2$  to  $x_1$ . Similarly, we get

$$(v_6, v_2) \rightarrow w_6.$$

This means that when individual A prefers  $x_2$  to  $x_3$  and individual B prefers  $x_3$  to  $x_2$ , then the society prefers  $x_2$  to  $x_3$ . Similarly, we get

$$(v_1, v_3) \rightarrow w_1.$$

This means that when individual A prefers  $x_1$  to  $x_3$  and individual B prefers  $x_3$  to  $x_1$ , then the society prefers  $x_1$  to  $x_3$ . Similarly, we get

$$(v_2, v_4) \rightarrow w_2.$$

This means that when individual A prefers  $x_1$  to  $x_2$  and individual B prefers  $x_2$  to  $x_1$ , then the society prefers  $x_1$  to  $x_2$ . These correspondences imply that individual A is the dictator. Therefore, if there is no dictator, we must have

$$(v_2, v_1) \rightarrow w_1.$$

This means that when individual A prefers  $x_3$  to  $x_2$  and individual B prefers  $x_2$  to  $x_3$ , then the society prefers  $x_2$  to  $x_3$ . By Pareto principle and independence of irrelevant alternatives we get

$$(v_3, v_1) \rightarrow w_1.$$

This means that when individual A prefers  $x_3$  to  $x_1$  and individual B prefers  $x_1$  to  $x_3$ , then the society prefers  $x_1$  to  $x_3$ . Similarly, we get

$$(v_4, v_2) \rightarrow w_2.$$

This means that when individual A prefers  $x_2$  to  $x_1$  and individual B prefers  $x_1$  to  $x_2$ , then the society prefers  $x_1$  to  $x_2$ . Then, by Pareto principle and independence of irrelevant alternatives we get correspondences from preference profiles to the social preference when the preference of individual B is fixed at  $v_1$  as follows:

$$(v_4, v_1) \rightarrow w_1, \quad (v_5, v_1) \rightarrow w_1, \quad (v_6, v_1) \rightarrow w_1.$$

Therefore, correspondences from the vertices of  $A$  to the vertices of  $S^1$  are obtained as follows:

$$\begin{aligned} (v_1, v_1) &\rightarrow w_1, & (v_2, v_1) &\rightarrow w_1, & (v_3, v_1) &\rightarrow w_1, \\ (v_4, v_1) &\rightarrow w_1, & (v_5, v_1) &\rightarrow w_1, & (v_6, v_1) &\rightarrow w_1. \end{aligned}$$

By similar logic, if individual B is not a dictator, correspondences from the vertices of  $B$  to the vertices of  $S^1$  are obtained as follows:

$$\begin{aligned} (v_1, v_1) &\rightarrow w_1, & (v_1, v_2) &\rightarrow w_1, & (v_1, v_3) &\rightarrow w_1, \\ (v_1, v_4) &\rightarrow w_1, & (v_1, v_5) &\rightarrow w_1, & (v_1, v_6) &\rightarrow w_1. \end{aligned}$$

Sets of simplices which are 1-dimensional cycles of  $\Delta \cup A \cup B$  are only the following  $z$  and its counterpart  $-z$ .

$$\begin{aligned} z = & \langle (v_1, v_1), (v_2, v_1) \rangle + \langle (v_2, v_1), (v_3, v_1) \rangle + \langle (v_3, v_1), (v_4, v_1) \rangle \\ & + \langle (v_4, v_1), (v_5, v_1) \rangle + \langle (v_5, v_1), (v_6, v_1) \rangle + \langle (v_6, v_1), (v_1, v_1) \rangle \\ & + \langle (v_1, v_1), (v_1, v_2) \rangle + \langle (v_1, v_2), (v_1, v_3) \rangle + \langle (v_1, v_3), (v_1, v_4) \rangle \\ & + \langle (v_1, v_4), (v_1, v_5) \rangle + \langle (v_1, v_5), (v_1, v_6) \rangle + \langle (v_1, v_6), (v_1, v_1) \rangle \\ & + \langle (v_1, v_1), (v_6, v_6) \rangle + \langle (v_6, v_6), (v_5, v_5) \rangle + \langle (v_5, v_5), (v_4, v_4) \rangle \\ & + \langle (v_4, v_4), (v_3, v_3) \rangle + \langle (v_3, v_3), (v_2, v_2) \rangle + \langle (v_2, v_2), (v_1, v_1) \rangle. \end{aligned}$$

Since  $\Delta \cup A \cup B$  has no 2-dimensional simplex,  $z$  is a representative element of homology classes of  $\Delta \cup A \cup B$ .  $z$  is transferred by the homomorphism of homology groups  $F_*$  induced by  $F$  to the following  $z'$  in  $S^1$ .

$$\begin{aligned}
z' &= \langle w_1, w_1 \rangle + \langle w_1, w_1 \rangle \\
&\quad + \langle w_1, w_1 \rangle \\
&\quad + \langle w_1, w_6 \rangle + \langle w_6, w_5 \rangle + \langle w_5, w_4 \rangle + \langle w_4, w_3 \rangle + \langle w_3, w_2 \rangle + \langle w_2, w_1 \rangle \\
&= \langle w_1, w_6 \rangle + \langle w_6, w_5 \rangle + \langle w_5, w_4 \rangle + \langle w_4, w_3 \rangle + \langle w_3, w_2 \rangle + \langle w_2, w_1 \rangle.
\end{aligned}$$

This is a cycle of  $S^1$ . Therefore, the homology group induced by  $(F_*)|_{A \cup A \cup B}$ , which is the homomorphism of homology groups induced by  $F|_{A \cup A \cup B}$ , is not trivial, and hence the degree of mapping of  $F|_{A \cup A \cup B}$  is not zero.  $\square$

From Theorem 1 we obtain the following result:

**Theorem 2.** *The non-existence of social welfare function which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator (the Arrow impossibility theorem) is equivalent to the Brouwer fixed point theorem.*

## 5. Concluding remarks

We have shown that with two individuals and three alternatives the Arrow impossibility theorem is equivalent to the Brouwer fixed point theorem on a 2-dimensional ball (circle) using elementary concepts and techniques of algebraic topology, in particular, homology groups of simplicial complexes, homomorphisms of homology groups.

Our approach may be applied to other social choice problems such as Wilson's impossibility theorem [14], the Gibbard–Satterthwaite theorem [7,12] and Amartya Sen's liberal paradox [13].

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