Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Mathematical Economics 43 (2007) 184–191
JOURNAL OF Mathematical ECONOMICS
www.elsevier.com/locate/jmateco

A topological approach to Wilson's impossibility theorem

Yasuhito Tanaka*

Faculty of Economics, Doshisha University, Kamigyo-ku, Kyoto 602-8580, Japan

Received 28 October 2004; received in revised form 22 August 2005; accepted 27 June 2006
Available online 3 January 2007

Abstract

We will present a topological approach to Wilson's impossibility theorem [Wilson, R.B., 1972. Social choice theory without the Pareto principle. Journal of Economic Theory 5, 478–486] that there exists no non-null binary social choice rule which satisfies transitivity, independence of irrelevant alternatives, non-imposition and has no dictator nor inverse dictator. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by [Baryshnikov, Y., 1993. Unifying impossibility theorems: a topological approach. Advances in Applied Mathematics 14, 404–415]. This paper extends the result about the Arrow impossibility theorem shown in [Tanaka, Y., 2006. A topological approach to the Arrow impossibility theorem when individual preferences are weak orders. Applied Mathematics and Computation 174, 961–981] to Wilson's theorem.
© 2006 Elsevier B.V. All rights reserved.

JEL classification: D71

Keywords: Wilson's impossibility theorem; Homology groups of simplicial complexes; Simplicial mappings

1. Introduction

Topological approaches to social choice problems have been initiated by Chichilnisky (1980). Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. This approach has been further developed by Chichilnisky (1979, 1982), Koshevoy (1997), Lauwers (2004), Weinberger (2004) and so on. On the other hand, Baryshnikov (1993, 1997) have presented a topological approach to the Arrow impossibility

* Fax: +81 75 251 3648.
E-mail address: yatanaka@mail.doshisha.ac.jp.

0304-4068/$ – see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.jmateco.2006.06.011

theorem (Arrow, 1963) in a discrete framework of social choice.[1] In this paper we will present a topological approach to Wilson's impossibility theorem (Wilson, 1972) that there exists no non-null binary social choice rule which satisfies transitivity, independence of irrelevant alternatives, non-imposition and has no dictator nor inverse dictator under the assumption of the *free triple property*. Our main tool is a homomorphism of homology groups of simplicial complexes induced by simplicial mappings.[2] This paper extends the result about the Arrow impossibility theorem shown in Tanaka (2006b) to Wilson's theorem. For other researches of topological approaches to social choice problems, see Tanaka (2006a,c,d).

In the next section we summarize our model and preliminary results about the homology groups of simplicial complexes which represent individual and social preferences according to Tanaka (2006b). In Section 3 we will prove the main results.

2. The model and simplicial complexes

There are $n(\geq 3)$ alternatives and $k(\geq 2)$ individuals. n and k are finite positive integers. Denote individual i's preference by p_i. A profile of individual preferences is denoted by $\mathbf{p}$, and the set of profiles is denoted by $\mathcal{P}^k$. The alternatives are represented by x_i, $i = 1, 2, \ldots, n$. Individual preferences over the alternatives are weak orders, that is, individuals strictly prefer one alternative to another, or are indifferent between them. We consider a binary social choice rule which determines a social preference corresponding to a profile. It is called a *social welfare function* and is denoted by $F(\mathbf{p})$. We assume the free triple property, that is, for each combination of three alternatives individual preferences are not restricted. If the society is indifferent about every pair of two alternatives, the social welfare function is called *null*. If a social welfare function is not null, that is, the social preference derived by the social welfare function is strict about at least one pair of alternatives, then the social welfare function is called *non-null*.

Social welfare functions must be non-null, and must satisfy transitivity, non-imposition and independence of irrelevant alternatives. The meanings of the latter two conditions are as follows.

Non-imposition For every pair of two alternatives x_i and x_j there exists a profile at which the society prefers x_i to x_j or is indifferent between them.

Independence of irrelevant alternatives (IIA) The social preference about any pair of two alternatives x_i and x_j is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x_i and x_j.

The impossibility theorem by Wilson (1972) states that there exists no non-null binary social choice rule which satisfies transitivity, IIA, non-imposition and has no dictator nor inverse dictator. A dictator is an individual whose strict preference always coincides with the social preference, and an inverse dictator is an individual whose strict preference always coincides with the inverse of the social preference.

Hereafter we will consider a set of alternatives x_1, x_2 and x_3. From the set of individual preferences about x_1, x_2 and x_3 we construct a simplicial complex by the following procedures.

[1] About surveys and basic results of topological social choice theories, see Mehta (1997) and Lauwers (2000).

[2] About homology groups we referred to Tamura (1970) and Komiya (2001).

(1) We denote a preference of an individual such that he prefers x_1 to x_2 by (1, 2), a preference such that he prefers x_2 to x_1 by (2, 1), a preference such that he is indifferent between x_1 and x_2 by $(\overline{1, 2})$, and similar for other pairs of alternatives. Define vertices of the simplicial complex corresponding to (i, j) and $(\overline{i, j})$.
(2) A line segment between the vertices (i, j) and (k, l) is included in the simplicial complex if and only if the preference represented by (i, j) and the preference represented by (k, l) satisfy transitivity. For example, the line segment between (1, 2) and (3, 2) is included, but the line segment between (1, 2) and (2, 1) is not included in the simplicial complex.
(3) A triangle (circumference plus interior) made by three vertices (i, j), (k, l) and (m, n) is included in the simplicial complex if and only if the preferences represented by (i, j), (k, l) and (m, n) satisfy transitivity. For example, a triangle made by (1, 2), (2, 3) and (1, 3) is included in the simplicial complex. But a triangle made by (1, 2), (2, 3) and (3, 1) is not included in the simplicial complex.

The simplicial complex constructed by these procedures is denoted by P. About a graphical presentation of the simplicial complexes see Tanaka (2006b).

We have shown the following result in Lemma 1 of Tanaka (2006b).

Lemma 1. *The one-dimensional homology group of* P *is isomorphic to the group of 6 integers, that is*, $H_1(P) \cong \mathbb{Z}^6$.

Also about the simplicial complex, P^k, made by the set of profiles of individual preferences, $\mathcal{P}^k$, over x_1, x_2 and x_3 we have shown the following result in Lemma 2 of Tanaka (2006b).

Lemma 2. *The one-dimensional homology group of* P^k *is isomorphic to the group of* $6k$ *integers, that is*, $H_1(P^k) \cong \mathbb{Z}^{6k}$.

The social preference about x_i and x_j is (i, j) or (j, i) or $(\overline{i, j})$, and it is also represented by P. By the condition of IIA, individual preferences about alternatives other than x_i and x_j do not affect the social preference about them. Thus, the social welfare function F is a function from the vertices of P^k to the vertices of P. A set of points in P^k spans a simplex if and only if individual preferences represented by these points satisfy transitivity, and then the social preference derived from the profile represented by these points also satisfies transitivity. Therefore, if a set of points in P^k spans a simplex, the set of points in P which represent the social preference corresponding to those points in P^k also spans a simplex in P, and hence the social welfare function is a *simplicial mapping*. It is naturally extended from the vertices of P^k to all points in P^k. Each point in P^k is represented as a convex combination of the vertices of P^k. This function is also denoted by F.

We define an inclusion mapping from P to P^k, $\Delta : P \longrightarrow P^k : p \longrightarrow (p, p, \dots, p)$, and an inclusion mapping which is derived by fixing the profile of preferences of individuals other than individual l to $\mathbf{p}_{-l}$, $i_l : P \longrightarrow P^k : p \longrightarrow (\mathbf{p}_{-l}, p)$. The homomorphisms of one-dimensional homology groups induced by these inclusion mappings are

$$\Delta_* : \mathbb{Z}^6 \longrightarrow \mathbb{Z}^{6k} : h \longrightarrow (h, h, \dots, h), \quad h \in \mathbb{Z}^6$$

$i_{l*} : \mathbb{Z}^6 \longrightarrow \mathbb{Z}^{6k} : h \longrightarrow (0, \dots, h, \dots, 0)$ (only the lth component is h and others are zero, $h \in \mathbb{Z}^6$)

From these definitions about Δ_* and i_{l*} we obtain the following relation:

$$\Delta_* = i_{1*} + i_{2*} + \cdots + i_{k*} \tag{1}$$

And the homomorphism of homology groups induced by F is represented as follows:

$$F_* : \mathbb{Z}^{6k} \longrightarrow \mathbb{Z}^6 : \mathbf{h} = (h_1, h_2, \ldots, h_k) \longrightarrow h, \quad h \in \mathbb{Z}^6$$

The composite function of i_l and the social welfare function F is $F \circ i_l : P \longrightarrow P$, and its induced homomorphism satisfies $(F \circ i_l)_* = F_* \circ i_{l*}$. The composite function of Δ and F is $F \circ \Delta : P \longrightarrow P$, and its induced homomorphism satisfies $(F \circ \Delta)_* = F_* \circ \Delta_*$. From (1) we have

$$(F \circ \Delta)_* = (F \circ i_1)_* + (F \circ i_2)_* + \cdots + (F \circ i_k)_*$$

$F \circ i_l$ when the profile of individuals other than individual l is $\mathbf{p}_{-l}$ and $F \circ i_l$ when the profile of individuals other than individual l is $(\neq \mathbf{p}_{-l})$ are homotopic. Thus, the induced homomorphism $(F \circ i_l)_*$ of $F \circ i_l$ does not depend on the preferences of individuals other than l.

For a pair of alternatives x_i and x_j, a profile, at which all individuals prefer x_i to x_j, is denoted by $(i, j)^{(+)}$; a profile, at which they prefer x_j to x_i, is denoted by $(i, j)^{(-)}$. And a profile, at which the preferences of all individuals about x_i and x_j are not specified, is denoted by $(i, j)^s$ where $s = \{+, 0, -\}^k$ with s_j the sign of individual j. 0 denotes a preference such that it is indifferent between x_i and x_j. Similarly a profile, at which all individuals other than l prefer x_i to x_j, is denoted by $(i, j)^{(+)}_{-l}$; a profile, at which they prefer x_j to x_i, is denoted by $(i, j)^{(-)}_{-l}$. And a profile, at which the preferences of individuals other than l about x_i and x_j are not specified, is denoted by $(i, j)^s_{-l}$.

3. The main results

First we show the following lemma.

Lemma 3. *If* $(F \circ \Delta)_* = 0$ *the society is indifferent about any pair of alternatives, that is, the social welfare function is null.*

Proof. Consider a set of three alternatives, x_1, x_2 and x_3. Assume that when all individuals prefer x_1 to x_2, the society prefers x_1 to x_2 (or x_2 to x_1), that is, assume the following correspondence from individual preferences to the social preference:

$$(1, 2)^{(+)} \longrightarrow (1, 2)\,[\text{or}\,(2, 1)]$$

By non-imposition there exists a profile such that we have the following correspondences:

$$(2, 3)^s \longrightarrow (2, 3)\ \text{or}\ (\overline{2, 3})\ [\text{or “}(3, 2)\ \text{or}\ (\overline{2, 3})\text{”}],$$

$$(1, 3)^s \longrightarrow (3, 1)\ \text{or}\ (\overline{1, 3})\ [\text{or “}(1, 3)\ \text{or}\ (\overline{1, 3})\text{”}]$$

Transitivity implies

$$(1, 3)^{(+)} \longrightarrow (1, 3)\,[\text{or}\,(3, 1)] \tag{2}$$

$$(2, 3)^{(-)} \longrightarrow (3, 2)\,[\text{or}\,(2, 3)] \tag{3}$$

Again, by non-imposition there exists a profile such that we have the correspondence:

$$(1, 2)^s \longrightarrow (2, 1)\ \text{or}\ (\overline{1, 2})\ [\text{or “}(1, 2)\ \text{or}\ (\overline{1, 2})\text{”}]$$

Then, from transitivity we obtain

$$(1, 3)^{(-)} \longrightarrow (3, 1)\,[\text{or}\,(1, 3)], \qquad (2, 3)^{(+)} \longrightarrow (2, 3)\,[\text{or}\,(3, 2)]$$

From these arguments we find that a cycle of P, $z = \langle(1,2),(2,3)\rangle + \langle(2,3),(3,1)\rangle - \langle(1,2),(3,1)\rangle$, corresponds to a cycle $z = \langle(1,2),(2,3)\rangle + \langle(2,3),(3,1)\rangle - \langle(1,2),(3,1)\rangle$, or a cycle $z' = \langle(2,1),(3,2)\rangle + \langle(3,2),(1,3)\rangle - \langle(2,1),(1,3)\rangle$ of P for the social preference by $(F \circ \Delta)_*$. Because both z and z' are not a boundary cycle, we have $(F \circ \Delta)_* \neq 0$. This result can be reached starting from an assumption other than $(1,2)^{(+)} \longrightarrow (1,2)$ [or $(1,2)^{(+)} \longrightarrow (2,1)$], for example, $(2,3)^{(+)} \longrightarrow (2,3)$ [or $(2,3)^{(+)} \longrightarrow (3,2)$].

Therefore, if $(F \circ \Delta)_* = 0$ we obtain the following correspondences from individual preferences to the social preference:

$$(1,2)^{(+)} \longrightarrow (\overline{1,2}), \quad (2,3)^{(+)} \longrightarrow (\overline{2,3}), \qquad (2,3)^{(-)} \longrightarrow (\overline{2,3}), \quad (1,3)^{(+)} \longrightarrow (\overline{1,3}) \tag{4}$$

From (4) with transitivity we obtain

$$(1,3)^s \longrightarrow (\overline{1,3}), \quad (2,3)^s \longrightarrow (\overline{2,3}), \quad (1,2)^s \longrightarrow (\overline{1,2})$$

Thus, the society is indifferent about any pair of alternatives among x_1, x_2 and x_3.

Interchanging x_3 with x_4 in the proof of this lemma, we can show that the society is indifferent about any pair of alternatives among x_1, x_2 and x_4. Similarly, the society is indifferent among x_5, x_2 and x_4, and it is indifferent among x_5, x_6 and x_4. After all the society is indifferent about any pair of alternatives, that is, the social welfare function is null. □

This lemma implies that if a social welfare function is non-null, we have $(F \circ \Delta)_* \neq 0$. Further we show the following lemma.

Lemma 4.

(1) *If individual l is a dictator or an inverse dictator, we have* $(F \circ i_l)_* \neq 0$.
(2) *If he is not a dictator nor inverse dictator, we have* $(F \circ i_l)_* = 0$.

Proof.

(1) Consider three alternatives x_1, x_2 and x_3. If individual l is a dictator, the correspondences from his preference to the social preference by $F \circ i_l$ are as follows:

$$(1,2)_l \longrightarrow (1,2), \quad (2,1)_l \longrightarrow (2,1), \quad (2,3)_l \longrightarrow (2,3), \qquad (3,2)_l \longrightarrow (3,2),$$
$$(1,3)_l \longrightarrow (1,3), \quad (3,1)_l \longrightarrow (3,1)$$

$(1,2)_l$ and $(2,1)_l$ denote the preference of individual l about x_1 and x_2. $(2,3)_l$, $(3,2)_l$ and so on are similar. These correspondences imply that a cycle of P, $z = \langle(1,2),(2,3)\rangle + \langle(2,3),(3,1)\rangle - \langle(1,2),(3,1)\rangle$, corresponds to the same cycle of P for the social preference by $(F \circ i_l)_*$. Because z is not a boundary cycle, we have $(F \circ i_l)_* \neq 0$.

On the other hand, if individual l is an inverse dictator, the correspondences from his preference to the social preference by $F \circ i_l$ are as follows:

$$(1,2)_l \longrightarrow (2,1), \quad (2,1)_l \longrightarrow (1,2), \quad (2,3)_l \longrightarrow (3,2), \qquad (3,2)_l \longrightarrow (2,3),$$
$$(1,3)_l \longrightarrow (3,1), \quad (3,1)_l \longrightarrow (1,3)$$

These correspondences imply that a cycle of P, $z = \langle(1,2),(2,3)\rangle + \langle(2,3),(3,1)\rangle - \langle(1,2),(3,1)\rangle$, corresponds to a cycle $z' = \langle(2,1),(3,2)\rangle + \langle(3,2),(1,3)\rangle - \langle(2,1),(1,3)\rangle$ of P for the social preference by $(F \circ i_l)_*$, and so we have $(F \circ i_l)_* \neq 0$.

(2) From the proof of Lemma 3 if a social welfare function is non-null, there are the following two cases:

Case (a) The following four correspondences simultaneously hold:

$$(1,2)^{(+)} \longrightarrow (1,2), \quad (1,3)^{(+)} \longrightarrow (1,3),$$
$$(2,3)^{(+)} \longrightarrow (2,3), \quad (2,3)^{(-)} \longrightarrow (3,2) \tag{5}$$

Case (b) The following four correspondences simultaneously hold:

$$(1,2)^{(+)} \longrightarrow (2,1), \quad (1,3)^{(+)} \longrightarrow (3,1),$$
$$(2,3)^{(+)} \longrightarrow (3,2), \quad (2,3)^{(-)} \longrightarrow (2,3) \tag{6}$$

We will provide the proof of Case (b). The proof of Case (a) is similar.

Consider three alternatives x_1, x_2 and x_3 and a profile $\mathbf{p}$ over them such that the preferences of individuals other than l are represented by $(1,2)^{(+)}_{-l}$, $(2,3)^{(+)}_{-l}$ and $(1,3)^{(+)}_{-l}$. If individual l is not an inverse dictator, there exists a profile at which the social preference about some pair of alternatives does not coincide with the inverse of his strict preference. Assume that when the preference of individual l is $(1,2)$, the social preference is $(1,2)$ or $(\overline{2,1})$. Then, we obtain the following correspondence from the profile to the social preference:

$$(1,2)^{s}_{-l} \times (1,2)_l \longrightarrow (1,2)\,\text{or}\,(\overline{1,2})$$

Then, from (6) and transitivity we obtain

$$(2,3)^{(+)}_{-l} \times (3,2)_l \longrightarrow (3,2)$$

and

$$(1,3)^{(+)}_{-l} \times (3,1)_l \longrightarrow (3,1)$$

Further, from (6) and transitivity we get the following correspondence:

$$(1,2)^{(+)}_{-l} \times (2,1)_l \longrightarrow (2,1)$$

These results imply that at a profile $\mathbf{p}$, where the preferences of individuals other than l are represented by $(1,2)^{(+)}_{-l}$, $(2,3)^{(+)}_{-l}$ and $(1,3)^{(+)}_{-l}$, the correspondences from the preference of individual l to the social preference by $F \circ i_l$ are as follows:

$$(1,2)_l \longrightarrow (2,1), \quad (2,1)_l \longrightarrow (2,1), \quad (2,3)_l \longrightarrow (3,2), \quad (3,2)_l \longrightarrow (3,2),$$
$$(1,3)_l \longrightarrow (3,1), \quad (3,1)_l \longrightarrow (3,1)$$

These correspondences with transitivity and IIA imply that, when individual l is indifferent between x_1 and x_3, the society prefers x_1 to x_3, that is, we obtain the following correspondence:

$$(\overline{1,3})_l \longrightarrow (3,1)$$

This is derived from two correspondences $(1,2)_l \longrightarrow (2,1)$ and $(3,2)_l \longrightarrow (3,2)$. Therefore, the following four sets of correspondences are impossible because the correspondences in each set are not consistent with the correspondence $(\overline{1,3})_l \longrightarrow (3,1)$:

(i) $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (\overline{2,3})$
(ii) $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (2,3)$
(iii) $(\overline{1,2})_l \longrightarrow (1,2),\ (\overline{2,3})_l \longrightarrow (2,3)$
(iv) $(\overline{1,2})_l \longrightarrow (1,2),\ (\overline{2,3})_l \longrightarrow (\overline{2,3})$

There are the following five cases, which are consistent with the correspondence $(\overline{1,3})_l \longrightarrow (3,1)$:

(i) Case (i): $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (3,2)$
(ii) Case (ii): $(\overline{1,2})_l \longrightarrow (2,1),\ (\overline{2,3})_l \longrightarrow (\overline{2,3})$
(iii) Case (iii): $(\overline{1,2})_l \longrightarrow (2,1),\ (\overline{2,3})_l \longrightarrow (3,2)$
(iv) Case (iv): $(\overline{1,2})_l \longrightarrow (1,2),\ (\overline{2,3})_l \longrightarrow (3,2)$
(v) Case (v): $(\overline{1,2})_l \longrightarrow (2,1),\ (\overline{2,3})_l \longrightarrow (2,3)$

We consider Case (i). The arguments for other cases are similar.

In Case (i) we have $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (3,2)$. The vertices of P for the social preference mapped from the preference of individual l by $F \circ i_l$ span the following five simplices:

$$\langle(2,1),(3,2)\rangle,\quad \langle(2,1),(3,1)\rangle,\quad \langle(3,2),(3,1)\rangle,\quad \langle(\overline{1,2}),(3,2)\rangle,\quad \langle(\overline{1,2}),(3,1)\rangle$$

Then, an element of the one-dimensional chain group is written as

$$c_1 = a_1\langle(2,1),(3,2)\rangle + a_2\langle(2,1),(3,1)\rangle + a_3\langle(3,2),(3,1)\rangle + a_4\langle(\overline{1,2}),(3,2)\rangle + a_5\langle(\overline{1,2}),(3,1)\rangle,\ a_i \in \mathbb{Z}$$

The condition for an element of the one-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2)\langle(2,1)\rangle + (a_1 - a_3 + a_4)\langle(3,2)\rangle + (a_2 + a_3 + a_5)\langle(3,1)\rangle + (-a_4 - a_5)\langle(\overline{1,2})\rangle = 0$$

From this

$$-a_1 - a_2 = 0,\ a_1 - a_3 + a_4 = 0,\ a_2 + a_3 + a_5 = 0,\ -a_4 - a_5 = 0$$

are derived. Then, we obtain $a_2 = -a_1$, $a_5 = -a_4$, $a_3 = a_1 + a_4$. Therefore, an element of the one-dimensional cycle group, Z_1, is written as follows:

$$z_1 = a_1\langle(2,1),(3,2)\rangle - a_1\langle(2,1),(3,1)\rangle + (a_1 + a_4)\langle(3,2),(3,1)\rangle + a_4\langle(\overline{1,2}),(3,2)\rangle - a_4\langle(\overline{1,2}),(3,1)\rangle$$

On the other hand, the vertices span the following two-dimensional simplices:

$$\langle(2,1),(3,2),(3,1)\rangle,\quad \langle(\overline{1,2}),(3,2),(3,1)\rangle$$

Then, an element of the two-dimensional chain group is written as

$$c_2 = b_1\langle(2,1),(3,2),(3,1)\rangle + b_2\langle(\overline{1,2}),(3,2),(3,1)\rangle,\ b_i \in \mathbb{Z}$$

And an element of the one-dimensional boundary cycle group, B_1, is written as follows:

$$\partial c_2 = b_1\langle(2,1),(3,2)\rangle - b_1\langle(2,1),(3,1)\rangle + (b_1 + b_2)\langle(3,2),(3,1)\rangle + b_2\langle(\overline{1,2}),(3,2)\rangle - b_2\langle(\overline{1,2}),(3,1)\rangle$$

Then, we find that B_1 is isomorphic to Z_1, and so the one-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

Thus, if there exists no inverse dictator, we have $(F \circ i_l)_* = 0$. □

In Case (a) we can show that if there exists no dictator, we have $(F \circ i_l)_* = 0$.

From these arguments and $(F \circ \Delta)_* \neq 0$ there exists a dictator or an inverse dictator about x_1, x_2 and x_3. Let individual l be a dictator or an inverse dictator. Interchanging x_3 with x_4 in the proof of this lemma, we can show that he is a dictator or an inverse dictator about x_1, x_2 and x_4. Similarly, we can show that he is a dictator or an inverse dictator about x_5, x_2 and x_4, he is a dictator or an inverse dictator about x_5, x_6 and x_4. After all he is a dictator or an inverse dictator about all alternatives.

From these lemmas we obtain the following theorem.

Theorem 1 (Wilson's impossibility theorem). *There exists a dictator or an inverse dictator for a social welfare function which is non-null, and satisfies IIA and non-imposition.*

Proof. From Lemma 3 if a social welfare function is non-null, we have $(F \circ \Delta)_* \neq 0$. Therefore, from Lemma 4 there exists a dictator or an inverse dictator. □

Acknowledgements

The author wishes to thank an anonymous referee for his (her) very valuable comments which have substantially improved the presentation of this paper. This research was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid in Japan.

References

Arrow, K.J., 1963. Social Choice and Individual Values, 2nd ed. Yale University Press.

Baryshnikov, Y., 1993. Unifying impossibility theorems: a topological approach. Advances in Applied Mathematics 14, 404–415.

Baryshnikov, Y., 1997. Topological and discrete choice: in search of a theory. Social Choice and Welfare 14, 199–209.

Chichilnisky, G., 1979. On fixed point theorems and social choice paradoxes. Economics Letters 3, 347–351.

Chichilnisky, G., 1980. Social choice and the topology. Advances in Mathematics 37, 165–176.

Chichilnisky, G., 1982. The topological equivalence of the Pareto condition and the existence of a dictator. Journal of Mathematical Economics 9, 223–233.

Komiya, K., 2001. Introduction to Topology. Shokabo (Isokika nyumon in Japanese).

Koshevoy, G., 1997. Homotopy properties of Pareto aggregation rules. Social Choice and Welfare 14, 295–302.

Lauwers, L., 2000. Topological social choice. Mathematical Social Sciences 40, 1–39.

Lauwers, L., 2004. Topological manipulators form an ultrafilter. Social Choice and Welfare 22, 437–445.

Mehta, P., 1997. Topological methods in social choice: An overview. Social Choice and Welfare 14, 233–243.

Tamura, I., 1970. Topology. Iwanami-shoten (in Japanese).

Tanaka, Y., 2006a. On the equivalence of the Arrow impossibility theorem and the Brouwer fixed point theorem. Applied Mathematics and Computation 172, 1303–1314.

Tanaka, Y., 2006b. A topological approach to the Arrow impossibility theorem when individual preferences are weak orders. Applied Mathematics and Computation 174, 961–981.

Tanaka, Y., 2006c. A topological proof of Eliaz's unified theorem of social choice theory. Applied Mathematics and Computation 176, 83–90.

Tanaka, Y., 2006d. On the topological equivalence of the Arrow impossibility theorem and Amartya Sen's liberal paradox, Applied Mathematics and Computation 181, 1490–1498.

Weinberger, S., 2004. On the topological social choice model. Journal of Economic Theory 115, 377–384.

Wilson, R.B., 1972. Social choice theory without the Pareto principle. Journal of Economic Theory 5, 478–486.