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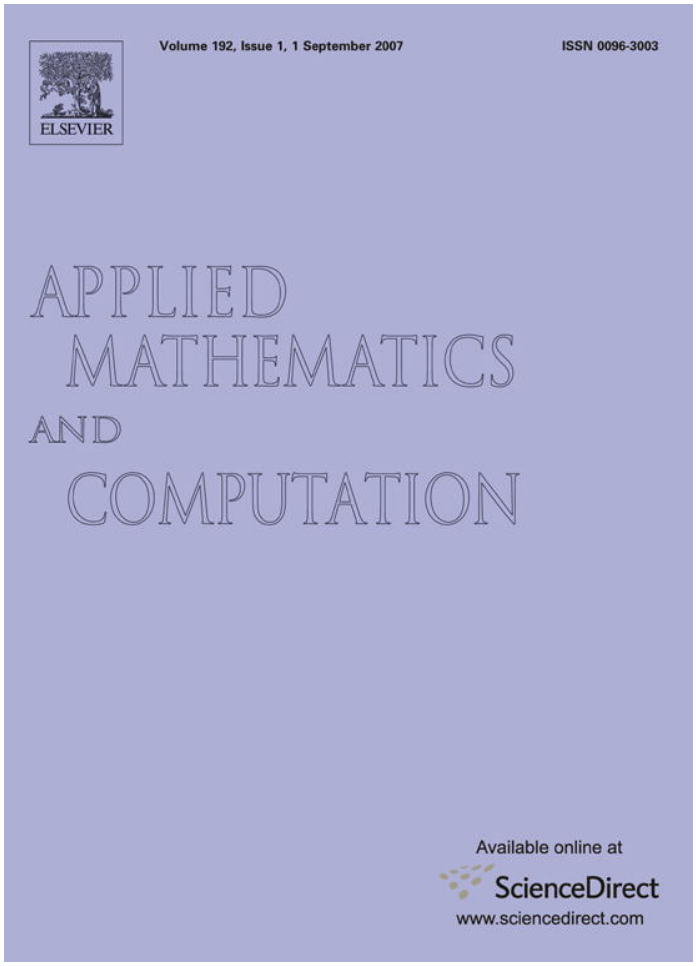

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Type two computability of social choice functions and the Gibbard–Satterthwaite theorem in an infinite society

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Abstract

This paper investigates the computability problem of the Gibbard–Satterthwaite theorem [A.F. Gibbard, Manipulation of voting schemes: a general result, Econometrica 41 (1973) 587–601; M.A. Satterthwaite, Strategyproofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions, Journal of Economic Theory 10 (1975) 187–217] of social choice theory in a society with an infinite number of individuals (infinite society) based on Type two computability by Weihrauch [K. Weihrauch, A Simple Introduction to Computable Analysis, Informatik Berichte, vol. 171, second ed., Fern Universität Hagen, Hagen, 1995; K. Weihrauch, Computable Analysis, Springer-Verlag, 2000]. There exists a dictator or there exists no dictator for any coalitionally strategy-proof social choice function in an infinite society. We will show that if there exists a dictator for a social choice function, it is computable in the sense of Type two computability, but if there exists no dictator it is not computable. A dictator of a social choice function is an individual such that if he strictly prefers an alternative (denoted by x) to another alternative (denoted by y), then it does not choose y, and his most preferred alternative is always chosen. Coalitional strategy-proofness is an extension of the ordinary strategy-proofness. It requires non-manipulability by coalitions of individuals as well as by a single individual.
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Keywords: Type two computability; Social choice function; The Gibbard–Satterthwaite theorem; Infinite society; Strategy-proofness; Dictator

1. Introduction

This paper investigates the computability problem of the Gibbard–Satterthwaite theorem [6,12] of social choice theory in a society with an infinite number of individuals (infinite society) based on Type two computability by Weihrauch [20,21].[2] Arrow's impossibility theorem [1] shows that, with a finite number of individuals, for any social welfare function (binary social choice rule which satisfies some conditions) there exists a

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[2] In other papers, [15–18], we have studied the computability problems of some other theorems of social choice theory such as the Arrow impossibility theorem, the theorem by Duggan and Schwartz [3] about the existence of dictator for strategy-proof social choice correspondences and the strong candidate stability theorem by Dutta et al. [4].

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dictator. In contrast Refs. [5,7,8] show that in a society with an infinite number of individuals (an infinite society), there exists a social welfare function without dictator. On the other hand, about strategy-proof social choice functions, with a finite number of individuals, the Gibbard–Satterthwaite theorem [6,12] shows that there exists a dictator for any strategy-proof social choice function. In contrast Ref. [11] shows that in an infinite society, there exists a coalitionally strategy-proof social choice function without dictator.[3] A dictator of a social choice function is an individual such that if he strictly prefers an alternative (denoted by x) to another alternative (denoted by y), then it does not choose y, and it chooses his most preferred alternative. Coalitional strategy-proofness is an extension of the ordinary strategy-proofness. It requires non-manipulability by coalitions of individuals as well as by a single individual.

In the next section we present the framework of this paper and some preliminary results. In Section 3 we will show the following results:

1. There exists a dictator or there exists no dictator for any coalitionally strategy-proof social choice function, and in the latter case all co-finite sets of individuals (sets of individuals whose complements are finite) are decisive sets (Theorem 1).
2. If there exists a dictator, the social choice function is computable in the sense of Type two computability, but if there exists no dictator it is not computable (Theorem 2).

A decisive set for a social choice function is a set of individuals such that if individuals in the set prefer an alternative (denoted by x) to another alternative (denoted by y), then the social choice function does not choose y regardless of the preferences of other individuals.

Mihara [9] presented an analysis about the ordinary Turing machine computability of social choice rules. Since there are only countable number of ordinary Turing machines, he assumes that only countable number of profiles of individual preferences are observable. But Type two machine can treat uncountable input.

2. The framework and preliminary results

There are m ($\geqslant 3$) alternatives and a countably infinite number of individuals. m is a finite positive integer. The set of alternatives is denoted by A. The set of individuals is denoted by N. The alternatives are represented by x, y, z, w and so on. Individual preferences over the alternatives are transitive linear (strict) orders, that is, they prefer one alternative to another alternative, and are not indifferent between them. Denote individual i's preference by $\succ_i$. We denote $x \succ_i y$ when individual i prefers x to y. Since there are a finite number of alternatives, the varieties of linear orders over the alternatives are finite. We denote the set of individual preferences by Σ. A combination of individual preferences, which is called a *profile*, is denoted by p $(= (\succ_1, \succ_2, \ldots))$, p' $(= (\succ'_1, \succ'_2, \ldots))$ and so on. The set of profiles is denoted by Σ^ω, where $\omega = \{1, 2, \ldots\}$ is the set of natural numbers. It represents the set of individuals.

We consider a social choice function $f : \Sigma^\omega \to A$ which chooses at least one and at most one alternative corresponding to each profile of the revealed preferences of individuals. We require that social choice functions are *coalitionally strategy-proof*. This means that any group (coalition) of individuals cannot benefit by revealing preferences which are different from their true preferences, in other words, each coalition of individuals must have an incentive to reveal their true preferences, and cannot manipulate any social choice function. The coalitional strategy-proofness is an extension of the ordinary strategy-proofness which requires only non-manipulability by an individual. We also require that social choice functions are *onto*, that is, their ranges are A. The Gibbard–Satterthwaite theorem states that, with a finite number of individuals, there exists a dictator for any strategy-proof social choice function, or in other words there exists no social choice function which satisfies strategy-proofness and has no dictator. In contrast Ref. [11] shows that when the number of individuals in the society is infinite, there exists a coalitionally strategy-proof social choice function without dictator. A dictator of a social choice function is an individual whose most preferred alternative is always chosen by the social choice function.

[3] [19] is a recent book that discusses social choice problems in an infinite society.

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According to Weihrauch [20,21] we survey the definitions of Type two machine and Type two computability, and consider the formulation of a social choice function computed by a Type two machine.

Type two machine. A Type two machine M (with one input tape) is defined by two components.

1. A Turing machine with a single one-way input tape, a single one-way output tape and finitely many work tapes.
2. A type specification (Y_1, Y_0) with $\{Y_1, Y_0\} \in \{\Sigma^*, \Sigma^\omega\}$. Σ denotes any finite alphabet. $\omega = \{1, 2, \ldots\}$ is the set of natural numbers. Σ^* is the set of all finite sequences $\succ_1 \succ_2 \cdots \succ_k$ with $k \in \omega$ and $\succ_1, \succ_2, \ldots, \succ_k \in \Sigma$. And $\Sigma^\omega = \{\succ_1 \succ_2 \cdots | \succ_i \in \Sigma\} = \{p | p : \omega \to \Sigma\}$ is the set of infinite sequences with elements from Σ. The function $f_M : Y_1 \to Y_0$ computed by a Type two machine M is defined as follows:
   (a) Case $Y_0 = \Sigma^*$ (finite output)
   $f_M(y_1) = w \in \Sigma^*$ if and only if M with input (y_1) halts with result w on the output tape.
   (b) Case $Y_0 = \Sigma^\omega$ (infinite output)
   $f_M(y_1) = p \in \Sigma^\omega$ if and only if M with input (y_1) computes forever writing the sequence p on the output tape.

Type two computability. Let Σ be a finite alphabet. Assume $Y_1 \subseteq \{\Sigma^*, \Sigma^\omega\}$. A function $f : Y_1 \to Y_0$ is computable if and only if $f = f_M$ for some Type two machine M.

A social choice function computed by a Type two machine. A social choice function is defined as a function $f : \Sigma^\omega \to A$. Σ^ω is the set of profiles, and A is the set of alternatives as alphabets. An element of Σ^ω, $p \in \Sigma^\omega$, is a profile, and an element of A is an alternative.

Now we define the following terms.[4]

Monotonicity. Let x and y be two alternatives. Assume that at a profile p individuals in a group G prefer x to y, all other individuals (individuals in $N \setminus G$) prefer y to x, and x is chosen by a social choice function. If at another profile p' individuals in G prefer x to y, then the social choice function does not choose y regardless of the preferences of the individuals in $N \setminus G$.

Weak Pareto principle. If all individuals prefer x to y, then any social choice function does not choose y.

First we can show the following lemma.

Lemma 1. *If a social choice function satisfies coalitional strategy-proofness, then it satisfies monotonicity and weak Pareto principle.*

Proof. See Appendix A. □

Further we define the following two terms:

Decisive. If, when all individuals in a group G prefer an alternative x to another alternative y, a social choice function does not chooses y regardless of the preferences of other individuals, then G is *decisive* for x against y.

Decisive set. If a group of individuals is decisive about every pair of alternatives for a social choice function, it is called a decisive set for the social choice function.

The meaning of the term *decisive* is similar to that of the same term used in [13] for binary social choice rules. G may consist of one individual. If for a social choice function an individual is decisive about every pair of alternatives, then he is a *dictator* of the social choice function.

About the concept of decisiveness we can show the following result.

Lemma 2. *Assume that a social choice function is coalitionally strategy-proof. If a group G is decisive for one alternative against another alternative, then it is a decisive set.*

Proof. See Appendix B. □

The implications of this lemma are similar to those of Lemma 3*a in [13] and Dictator Lemma in [14] for binary social choice rules.

[4] The concept of *monotonicity* is according to Batteau et al. [2]. It is equivalent to *strong positive association* by Muller and Satterthwaite [10] when individual preferences are linear orders (do not include indifference relations).

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Next we can show the following result.

Lemma 3. *Assume that a social choice function is coalitionally strategy-proof. If two groups G and G', which are not disjoint, are decisive sets, then their intersection $G \cap G'$ is a decisive set.*

Proof. Let x, y and z be given three alternatives, and consider the following profile:

1. Individuals in $G \setminus (G \cap G')$ prefer z to x to y to all other alternatives.
2. Individuals in $G' \setminus (G \cap G')$ prefer y to z to x to all other alternatives.
3. Individuals in $G \cap G'$ prefer x to y to z to all other alternatives.
4. Individuals in $N \setminus (G \cup G')$ prefer z to y to x to all other alternatives.

Since G and G' are decisive sets, the social choice function chooses x. Only individuals in $G \cap G'$ prefer x to z and all other individuals prefer z to x. Thus, by monotonicity $G \cap G'$ is decisive for x against z. By Lemma 2 it is a decisive set. □

Note that G and G' cannot be disjoint. Assume that G and G' are disjoint. If individuals in G prefer x to y to all other alternatives, and individuals in G' prefer y to x to all other alternatives, then neither G nor G' can be a decisive set.

This lemma implies that the intersection of a finite number of decisive sets is also a decisive set.

3. Type two computability of coalitionally strategy-proof social choice functions

Consider profiles such that one individual (denoted by i) prefers x to y to z to all other alternatives, and all other individuals prefer z to x to y to all other alternatives. Denote such a profile by p^i. By weak Pareto principle any social choice function chooses x or z. If a social choice function chooses x at p^i for some i, then by monotonicity individual i is decisive for x against z, and by Lemma 2 he is a dictator. On the other hand, if the social choice function chooses z at p^i for all $i \in N$, then there exists no dictator, and a group $N \setminus \{i\}$ is a decisive set for all $i \in N$. By Lemma 3 in the latter case all co-finite sets (sets of individuals whose complements are finite sets) are decisive sets. Thus, we obtain the following theorem.

Theorem 1. *For any coalitionally strategy-proof social choice function there exists a dictator or there exists no dictator, and in the latter case all co-finite sets are decisive sets.*

Let partition the set of individuals into a finite number of groups $G_1, G_2, \ldots, G_k$. Each group may include a finite or an infinite number of individuals. If there exists no dictator, all co-finite sets are decisive sets, and then every finite group is not a decisive set. Therefore, the decisive set must be an infinite group, and we obtain the following result.

Lemma 4. *Suppose that there exists no dictator for a social choice function. Let partition the set of individuals into a finite number of groups $G_1, G_2, \ldots, G_k$. Each group may include a finite or an infinite number of individuals. Then, one of infinite groups is a decisive set.*

Finally we show the following main result of this paper.

Theorem 2

1. *If there exists a dictator for a social choice function, then it is computable in the sense of Type two computability.*
2. *If there exists no dictator for a social choice function, then it is not computable.*

Proof

1. Assume that individual i $(\in N)$ is a dictator of a social choice function. A Type two machine can determine the choice of the society from the ith input, and then it halts.

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2. Let partition the individuals into a finite number of groups corresponding to the preferences of individuals in each group. Consider a profile $p \in \Sigma^{\omega}$ such that in such a partition only one group includes an infinite number of individuals. Then, by Lemma 4 this group is a decisive set. But any Type two machine cannot determine which group is an infinite group in finite steps, and it cannot halt. Therefore, the social choice function is not computable. □

4. Concluding remarks

We have examined the computability problem of the Gibbard–Satterthwaite theorem of social choice theory in an infinite society. The assumption of an infinite society seems to be unrealistic. But Mihara [9] presented an interpretation of an infinite society based on a *finite* number of individuals and a countably infinite number of uncertain states.

In this paper we assumed that individual preferences are linear orders, that is, they are not indifferent about any pair of alternatives. In the case of weak orders, which include indifference relations, a social choice function may not be computable even when there exists a dictator. Consider a social choice function such that when its dictator's most preferred alternatives are not unique, the society's choice is determined by preferences of a group with an infinite number of individuals. Then this social choice function is not computable.

Appendix A. Proof of Lemma 1

We use notations in the definition of monotonicity.

1. *Monotonicity*. Let z be an arbitrary alternative other than x and y. Assume that at a profile p'' individuals in G prefer x to y to z, and other individuals prefer y to x to z. If, when the preferences of some individuals in G change from $\succ_i$ (their preferences at p) to $\succ_i''$ (their preferences at p''), x is not chosen by the social choice function, then they can gain benefit by revealing their preferences $\succ_i$ when their true preferences are $\succ_i''$. Thus, the social choice function continues to choose x in this case. By the same logic, when the preferences of all individuals in G change to their preferences at p'', the social choice function chooses x. Next, if, when the preferences of some individuals in $N \setminus G$ change from $\succ_i$ to $\succ_i''$, the social choice function chooses y, then they can gain benefit by revealing their preferences $\succ_i''$ when their true preferences are $\succ_i$. On the other hand, if z is chosen in this case, they can gain benefit by revealing their preferences $\succ_i$ when their true preferences are $\succ_i''$. Thus, x must be chosen. By the same logic, when the preferences of all individuals change to their preferences at p'', the social choice function chooses x.

   Next, if, when the preferences of some individuals in G change from $\succ_i''$ to $\succ_i'$ (their preferences at p'), the alternative chosen by the social choice function changes directly from x to y, then they can gain benefit by revealing their preferences $\succ_i''$ when their true preferences are $\succ_i'$. Thus, the alternative chosen by the social choice function does not directly change from x to y in this case. By the same logic, when the preferences of all individuals in G change to their preferences at p', the alternative chosen by the social choice function does not directly change from x to y. Further, if, when the preferences of some individuals in $N \setminus G$ change from $\succ_i''$ to $\succ_i'$, the alternative chosen by the social choice function changes directly from x to y, then they can gain benefit by revealing their preference $\succ_i'$ when their true preferences are $\succ_i''$. By the same logic, when the preferences of all individuals change to their preferences at p', the alternative chosen by the social choice function does not directly change from x to y.

   There is a possibility, however, that the alternative chosen by the social choice function changes from x to w $(\neq x, y)$ to y in transition from p'' to p'. If, when the preferences of some individuals change, the alternative chosen by the social choice function changes from x to w, and further when the preferences of other some individuals (denoted by i) change, the alternative chosen by the social choice function changes to y, they have incentives to reveal their preferences $\succ_i'$ when their true preferences are $\succ_i''$ because they prefer y to w at p''. Therefore, y is not chosen by the social choice function at p'.
2. *Weak Pareto principle*. Let p be a profile at which all individuals prefer x to y, and p' be a profile at which x is chosen by the social choice function. Assume that at another profile p'' all individuals prefer x to y to all

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other alternatives. If, when the preferences of some individuals change from $\succ_i'$ to $\succ_i''$, the social choice function does not choose x, then they can gain benefit by revealing their preferences $\succ_i'$ when their true preferences are $\succ_i''$. Thus, x is chosen in this case. By the same logic, when the preferences of all individuals change to their preferences at p'', x is chosen. Since at p'' and at p all individuals prefer x to y, monotonicity (proved in 1) implies that y is not chosen by the social choice function at p.

Appendix B. Proof of Lemma 2

1. Case 1: There are more than three alternatives.
   Assume that G is decisive for x against y. Let z and w be given alternatives other than x and y. Consider the following profile:
   (a) Individuals in G prefer z to x to y to w to all other alternatives.
   (b) Other individuals prefer y to w to z to x to all other alternatives.
   By weak Pareto principle the social choice function chooses y or z. Since G is decisive for x against y, z is chosen. Then, by monotonicity the social choice function does not choose w so long as the individuals in G prefer z to w. It means that G is decisive for z against w. From this result by similar procedures we can show that G is decisive for x (or y) against w, for z against x (or y), and for y against x. Since z and w are arbitrary, G is decisive about every pair of alternatives, that is, it is a decisive set.
2. Case 2: There are only three alternatives x, y and z.
   Assume that G is decisive for x against y. Consider the following profile:
   (a) Individuals in G prefer x to y to z.
   (b) Other individuals prefer y to z to x.
   By weak Pareto principle the social choice function chooses x or y. Since G is decisive for x against y, x is chosen. Then, by monotonicity the social choice function does not choose z so long as the individuals in G prefer x to z. It means that G is decisive for x against z. Similarly we can show that G is decisive for z against y considering the following profile:
   (a) Individuals in G prefer z to x to y.
   (b) Other individuals prefer y to z to x.
   By similar procedures we can show that G is decisive for y against z, for z against x, and for y against x.

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