On the topological equivalence of the Arrow impossibility theorem and Amartya Sen’s liberal paradox

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Abstract

We will show that the Arrow impossibility theorem for binary social choice rules that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives (IIA), and has no dictator, and Amartya Sen’s liberal paradox for binary social choice rules that there exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism are topologically equivalent using elementary tools of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by [Y. Baryshnikov, Unifying impossibility theorems: a topological approach, Advances in Applied Mathematics 14 (1993) 404–415]. Also we will show that these two theorems are special cases of the theorem that there exists no binary social choice rule which satisfies Pareto principle and the non-surjectivity of individual inclusion mappings.

Keywords: Simplicial complexes; Simplicial mappings; Homology groups; Homomorphisms; Arrow impossibility theorem; Amartya Sen’s liberal paradox

1. Introduction

Topological approaches to social choice problems have been initiated by Chichilnisky [6]. Her main result is an impossibility theorem that there exists no continuous social choice rule which satisfies unanimity and anonymity. This approach has been further developed by Chichilnisky [5,7], Candeal and Indurain [4], Koshevoy [9], Lauwers [11], Weinberger [15], and so on. On the other hand, Baryshnikov [2,3] have presented a topological approach to the Arrow impossibility theorem (or general possibility theorem) in a discrete framework of social choice.1

We will show that the Arrow impossibility theorem for binary social choice rules that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives (IIA), and has no dictator, and Amartya Sen’s liberal paradox for binary social choice rules that there exists no
binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism are topologically equivalent using elementary tools of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings. Also we will show that these two theorems are special cases of the theorem that there exists no binary social choice rule which satisfies Pareto principle and the non-surjectivity of individual inclusion mappings. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by Baryshnikov [2].

In the next section we present expressions of binary social choice rules by simplicial complexes and simplicial mappings. In Section 3, we will prove the main results of this paper.

2. The expressions of social choice problems by simplicial complexes and simplicial mappings

There are $m$ alternatives of a social problem, $x_1, x_2, \ldots, x_m$ ($m \geq 3$), and $n$ individuals ($n \geq 2$). $m$ and $n$ are finite integers. Individual preferences over these alternatives are complete, transitive and asymmetric.

A social choice rule which we will consider is a rule that determines a social preference about each pair of alternatives corresponding to a combination of individual preferences. We call such a social choice rule a binary social choice rule. The social preference should be complete, but may be or may not be transitive. As usual we assume the universal domain condition for social choice rules. We call a combination of individual preferences a profile. The profiles are denoted by $p, p'$ and so on.

We will consider two social choice problems about binary social choice rules.

(1) (Amartya Sen’s liberal paradox): The liberal paradox by Amartya Sen [13] states that there exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism. The means of these conditions are as follows.

**Acyclicity:** If the society (strictly) prefers $x_i$ to $x_j$, and (strictly) prefers $x_j$ to $x_k$, then it should prefer $x_i$ to $x_k$ or be indifferent between them. It is weaker than transitivity which requires that the society (strictly) prefers $x_i$ to $x_k$.

**Pareto principle:** If all individuals prefer an alternative $x_i$ to another alternative $x_j$, then the society prefers $x_i$ to $x_j$.

**Minimal liberalism:** At least two individuals, denoted by A and B, are decisive for some pairs of alternatives in both directions in the sense described in the following Assumption 1.

In what follows as the condition of the minimal liberalism we assume

**Assumption 1.** If individual A prefers $x_1$ to $x_4$ (or prefers $x_4$ to $x_1$), then the society prefers $x_1$ to $x_4$ (or prefers $x_4$ to $x_1$). And if individual B prefers $x_2$ to $x_3$ (or prefers $x_3$ to $x_2$), then the society prefers $x_2$ to $x_3$ (or prefers $x_3$ to $x_2$).

Other individuals are not necessarily decisive. We can proceed the arguments in a similar manner based on other assumptions about the minimal liberalism by permuting or renaming alternatives. We abbreviate the problem of the liberal paradox as LP.

(2) (The Arrow impossibility theorem): The Arrow impossibility theorem [1] states that there exists no binary social choice rule which satisfies transitivity, Pareto principle and independence of irrelevant alternatives (IIA), and has no dictator, or in other words there exists the dictator for any binary social choice rule which satisfies transitivity, Pareto principle and IIA. The dictator for a binary social choice rule is an individual such that whenever he (strictly) prefers one alternative (denoted by $x$) to another alternative (denoted by $y$), the society also (strictly) prefers $x$ to $y$. The meanings of two conditions, transitivity and IIA, are as follows.

**Transitivity:** If the society (strictly) prefers $x_i$ to $x_j$, and (strictly) prefers $x_j$ to $x_k$, then the society should (strictly) prefer $x_i$ to $x_k$.

**Independence of irrelevant alternatives (IIA):** The society’s preference about any pair of two alternatives depends only on individual preferences about these alternatives.
We abbreviate the problem of the Arrow impossibility theorem as AR. Pareto principle for AR is the same as that for LP.

We draw a circumference which represents the set of individual preferences by connecting $m!$ vertices $v_1, v_2, \ldots, v_m$ by arcs.\footnote{m! denotes the factorial of $m$.} For example, in the case of four alternatives, these vertices mean the following preferences:

\begin{align*}
&v_1 : (1234), \quad v_2 : (1243), \quad v_3 : (1423), \quad v_4 : (1432), \quad v_5 : (1342), \quad v_6 : (1324), \\
&v_7 : (2134), \quad v_8 : (2143), \quad v_9 : (2413), \quad v_{10} : (2431), \quad v_{11} : (2341), \quad v_{12} : (2314), \\
&v_{13} : (3124), \quad v_{14} : (3142), \quad v_{15} : (3412), \quad v_{16} : (3421), \quad v_{17} : (3241), \quad v_{18} : (3214), \\
&v_{19} : (4123), \quad v_{20} : (4132), \quad v_{21} : (4312), \quad v_{22} : (4321), \quad v_{23} : (4231), \quad v_{24} : (4213).
\end{align*}

We denote a preference such that an individual prefers $x_1$ to $x_2$ to $x_3$ to $x_4$ by $(1234)$, and so on. Notations for the cases with different number of alternatives are similar. Generally $v_1 \sim v_{(m-1)!}$ represent preferences such that the most preferred alternative for an individual is $x_1$, $v_{(m-1)!+1} \sim v_{2(m-1)!}$ represent preferences such that the most preferred alternative for an individual is $x_2$, and so on. And $v_1$ is a preference such that an individual prefers $x_1$ to $x_2$ to $x_3$ to $\cdots$ to $x_m$. It is denoted by $(123 \cdots m)$. $v_{(m-1)!+1}$ is a preference such that an individual prefers $x_2$ to $x_1$ to $x_3$ to $x_4$ to $\cdots$ to $x_m$, which is denoted by $(2134 \cdots m)$.

Denote this circumference by $S_1^1$. In the case of three alternatives is depicted in Fig. 1. The set of profiles of the preferences of $n$ individuals is represented by the product space $S_1^1 \times \cdots \times S_1^1$ ($n$ times). It is denoted by $(S_1^1)^n$. The one-dimensional homology group of $S_1^1$ is isomorphic to the group of integers $\mathbb{Z}$, that is, $H_1(S_1^1) \cong \mathbb{Z}$. And the one-dimensional homology group of $(S_1^1)^n$ is isomorphic to the direct product of $n$ groups of integers $\mathbb{Z}^n$, that is, we have $H_1((S_1^1)^n) \cong \mathbb{Z}^n$. It is proved, for example, using the Mayer–Vietoris exact sequences.\footnote{About homology groups and the Mayer–Vietoris exact sequences we referred to Tamura [14] and Komiya [8].}

The social preference is also represented by a circumference depicted in Fig. 2. This circumference is drawn by connecting three vertices, $w_1$, $w_2$ and $w_3$ by arcs. For LP these vertices mean the following social preferences:

1. $w_1$: social preferences such that the society prefers $x_4$ to all other alternatives,
2. $w_2$: social preferences such that the society prefers $x_3$ to all other alternatives,
3. $w_3$: all other social preferences.
Similarly for AR these vertices mean the following social preferences:

1. \( w_1 \): social preferences such that the society prefers \( x_4 \) to all other alternatives,
2. \( w_3 \): social preferences such that the society prefers \( x_3 \) to all other alternatives,
3. \( w_2 \): all other social preferences.

That is, the vertices \( w_1 \) and \( w_3 \) denote the same social preferences for LP and AR, and the set of social preferences expressed by \( w_2 \) for AR is the proper subset of the set of social preferences expressed by \( w_2 \) for LP because the social preference are required to satisfy transitivity in AR, but in LP we require only acyclicity.

We call this circumference \( S^1 \). The one-dimensional homology group of \( S^1 \) is also isomorphic to \( \mathbb{Z} \), that is, \( H_1(S^1) \cong \mathbb{Z} \).

**Binary social choice rules are simplicial mappings.** Binary social choice rules in AR and LP are denoted by \( f : (S^1_i)^n \to S^1 \). Two adjacent vertices of \( S^1_i \) span a one-dimensional simplex. And any pair of two vertices of \( S^1 \) spans a one-dimensional simplex. Thus, \( f \) is a simplicial mapping, and we can define the homomorphism of homology groups induced by \( f \).

We define an inclusion mapping from \( S^1_i \) to \((S^1_i)^n \) by \( A : S^1_i \to (S^1_i)^n \) under the assumption that all individuals have the same preferences, and define an inclusion mapping when the profile of preferences of individuals other than one individual (denoted by \( i \)) is fixed at some profile by \( i : S^1_i \to (S^1_i)^n \). The homomorphisms of homology groups induced by these inclusion mappings are as follows:

\[
\begin{align*}
A_* : \mathbb{Z} &\to \mathbb{Z}^n : h \to (h, h, \ldots, h), \quad h \in \mathbb{Z}, \\
i_* : \mathbb{Z} &\to \mathbb{Z}^n : h \to (0, \ldots, 0, h, 0, \ldots, 0), \quad h \in \mathbb{Z} \text{ (only the } i\text{th component is } h). 
\end{align*}
\]

From these definitions we obtain the following relation about \( A_* \) and \( i_* \) at any profile:

\[
A_* = \sum_{i=1}^n i_*.
\]

Let the homomorphism of homology groups induced by \( f \) be \( f_* : (\mathbb{Z})^n \to \mathbb{Z} \).

**Binary social choice rules for different profiles are homotopic.** \( f \) for a fixed profile of the preferences of individuals other than \( i \) (denoted by \( f|_{p_{-i}} \)) and \( f \) for another fixed profile of the preferences of individuals other than \( i \) (denoted by \( f|_{p'_{-i}} \)) are homotopic. Thus, the homomorphisms of homology groups induced by them are isomorphic. Denote two profiles of individuals other than \( i \) by \( p_{-i} \) and \( p'_{-i} \). Then, the homotopy between \( f|_{p_{-i}} \) and \( f|_{p'_{-i}} \) is \( f_t = (1-t)f|_{p_{-i}} + tf|_{p'_{-i}} \) \((0 \leq t \leq 1)\). It is well defined since \( f|_{p_{-i}} \) and \( f|_{p'_{-i}} \) are not antipodal.
The composite function of \( i_i \) and \( f \) is denoted by \( f \circ i_i : S_i^1 \rightarrow S_i^1 \), and its induced homomorphism of homology groups satisfies \( (f \circ i_i)_* = f_* \circ i_\ast \) for all \( i \). The composite function of \( \Delta \) and \( f \) is denoted by \( f \circ \Delta : S_i^1 \rightarrow S_i^1 \), and its induced homomorphism of homology groups satisfies \( (f \circ \Delta)_* = f_* \circ \Delta_* \). From (1) we obtain

\[
(f \circ \Delta)_* = \sum_{i=1}^{n} (f \circ i_i)_*. 
\]

3. The main results

For binary social choice rules in AR we define the following concept:

**Weak monotonicity.** For two alternatives \( x_i \) and \( x_j \), suppose that at profile \( p \) the society prefers \( x_i \) to \( x_j \) and suppose that individuals, who prefer \( x_i \) to \( x_j \) at \( p \), prefer \( x_i \) to \( x_j \) at another profile \( p' \). Then, the society prefers \( x_i \) to \( x_j \) at \( p' \).

**Proof.** We use notations in the definition of the weak monotonicity. Let \( x_k \) be an arbitrary alternative other than \( x_i \) and \( x_j \).

Suppose that individuals, who prefer \( x_i \) to \( x_j \) at \( p \), prefer \( x_i \) to \( x_j \) at another profile \( p'' \), and individuals, who prefer \( x_i \) to \( x_j \) at \( p \), prefer \( x_i \) to \( x_k \) to \( x_j \) at \( p'' \).

And suppose that individuals, who prefer \( x_i \) to \( x_j \) at \( p \), prefer \( x_i \) to \( x_k \) to \( x_j \) at \( p'' \), and individuals, who prefer \( x_j \) to \( x_i \) at \( p \), prefer \( x_k \) to \( x_i \) and prefer \( x_k \) to \( x_j \) at \( p'' \) (their preferences about \( x_i \) and \( x_j \) are not specified).

By transitivity, Pareto principle and IIA the society prefers \( x_i \) to \( x_j \) at \( p'' \). Again by transitivity, Pareto principle and IIA (about \( x_i \) and \( x_k \)) the society prefers \( x_i \) to \( x_k \) to \( x_j \) at \( p'' \). Then, IIA implies that the society prefers \( x_i \) to \( x_j \) so long as individuals, who prefer \( x_i \) to \( x_j \) at \( p \), prefer \( x_i \) to \( x_j \) at an arbitrary profile \( p' \).

Next we show the following lemma which will be used below:

**Lemma 1.** Any binary social choice rule in AR which satisfies transitivity, Pareto principle and IIA satisfies the weak monotonicity.

**Proof.** We use notations in the definition of the weak monotonicity. Let \( x_k \) be an arbitrary alternative other than \( x_i \) and \( x_j \).

Suppose that individuals, who prefer \( x_i \) to \( x_j \) at \( p \), prefer \( x_i \) to \( x_j \) at another profile \( p'' \), and individuals, who prefer \( x_i \) to \( x_j \) at \( p \), prefer \( x_i \) to \( x_k \) to \( x_j \) at \( p'' \).

And suppose that individuals, who prefer \( x_i \) to \( x_j \) at \( p \), prefer \( x_i \) to \( x_k \) to \( x_j \) at \( p'' \), and individuals, who prefer \( x_j \) to \( x_i \) at \( p \), prefer \( x_k \) to \( x_i \) and prefer \( x_k \) to \( x_j \) at \( p'' \) (their preferences about \( x_i \) and \( x_j \) are not specified).

By transitivity, Pareto principle and IIA the society prefers \( x_i \) to \( x_j \) at \( p'' \). Again by transitivity, Pareto principle and IIA (about \( x_i \) and \( x_k \)) the society prefers \( x_i \) to \( x_k \) to \( x_j \) at \( p'' \). Then, IIA implies that the society prefers \( x_i \) to \( x_j \) so long as individuals, who prefer \( x_i \) to \( x_j \) at \( p \), prefer \( x_i \) to \( x_j \) at an arbitrary profile \( p' \).

**Lemma 2.** Suppose that a binary social choice rule satisfies transitivity, Pareto principle, IIA, and has no dictator. If the preference of one individual (denoted by \( i \)) is \( v_{(m-1)t+1} \) and the preferences of all other individuals are \( v_1 \), then the most preferred alternative for the society is \( x_1 \).

**Proof.** Note that \( v_{(m-1)t+1} \) represents a preference \((123 \ldots m)\) and \( v_1 \) represents a preference \((123 \ldots m)\). By Pareto principle the society prefers \( x_1 \) and \( x_2 \) to all other alternatives. It may prefer \( x_1 \) to \( x_2 \), or \( x_2 \) to \( x_1 \). But we can show that if the society prefers \( x_2 \) to \( x_1 \), individual \( i \) is the dictator. Assume that the society prefers \( x_2 \) to \( x_1 \) to all other alternatives. By the weak monotonicity the society prefers \( x_2 \) to \( x_1 \) so long as individual \( i \) prefers \( x_2 \) to \( x_1 \). Then, we say that individual \( i \) is decisive for \( x_2 \) against \( x_1 \). Let \( x_j \) and \( x_k \) \((x_k \neq x_j)\) be alternatives other than \( x_1 \) and \( x_2 \), and consider the following profile:

1. Individual \( i \) prefers \( x_k \) to \( x_2 \) to \( x_1 \) to \( x_j \).
2. Other individuals prefer \( x_1 \) to \( x_j \) to \( x_k \) to \( x_2 \).

By the weak monotonicity (or IIA) the society should prefer \( x_2 \) to \( x_1 \). And by Pareto principle the society should prefer \( x_1 \) to \( x_j \) and prefer \( x_k \) to \( x_2 \). Then, transitivity implies that the society prefers \( x_k \) to \( x_j \). The weak monotonicity

\[ \text{From Lemma 1 of Baryshnikov [2] we know that if individual preferences are strict orders, then the social preference is also a strict order under transitivity, Pareto principle and IIA.} \]
monotonicity implies that the society prefers $x_k$ to $x_j$ so long as individual $i$ prefers $x_k$ to $x_j$, and individual $i$ is decisive for $x_k$ against $x_j$. Note that $x_j$ and $x_k$ are arbitrary. Next consider the following profile:

(1) Individual $i$ prefers $x_2$ to $x_k$ to $x_j$.
(2) Other individuals prefer $x_j$ to $x_2$ to $x_k$.

By the weak monotonicity the society should prefer $x_k$ to $x_j$. And by Pareto principle the society should prefer $x_2$ to $x_k$. Then, transitivity implies that the society prefers $x_j$ to $x_2$. The weak monotonicity implies that the society prefers $x_2$ to $x_j$ so long as individual $i$ prefers $x_2$ to $x_j$, and individual $i$ is decisive for $x_2$ against $x_j$.

Consider the following profile:

(1) Individual $i$ prefers $x_k$ to $x_j$ to $x_2$.
(2) Other individuals prefer $x_j$ to $x_2$ to $x_k$.

By the weak monotonicity the society should prefer $x_k$ to $x_j$. And by Pareto principle the society should prefer $x_2$ to $x_k$. Then, transitivity implies that the society prefers $x_j$ to $x_2$. The weak monotonicity implies that the society prefers $x_2$ to $x_j$ so long as individual $i$ prefers $x_2$ to $x_j$, and individual $i$ is decisive for $x_2$ against $x_j$.

By similar procedures we can show that individual $i$ is decisive for $x_1$ against $x_j$, and is decisive for $x_k$ against $x_1$. Finally consider the following profile:

(1) Individual $i$ prefers $x_1$ to $x_k$ to $x_j$.
(2) Other individuals prefer $x_j$ to $x_1$ to $x_k$.

By the weak monotonicity the society should prefer $x_k$ to $x_j$. And by Pareto principle the society should prefer $x_1$ to $x_k$. Then, transitivity implies that the society prefers $x_j$ to $x_1$. The weak monotonicity implies that the society prefers $x_1$ to $x_j$ so long as individual $i$ prefers $x_1$ to $x_j$, and individual $i$ is decisive for $x_1$ against $x_j$. Therefore, individual $i$ is the dictator, and we must assume that the society prefers $x_1$ to all other alternatives when the preference of individual $i$ is $v_{(m-1)!+1}$ and the preferences of individuals other than $i$ are $v_1$.

In both AR and LP cases, by Pareto principle we obtain the correspondences from the vertices of $S^1_i$ to the vertices of $S^1_j$ by $f \circ A$ as follows:

$$w_1 \sim v_{2(m-1)!} \rightarrow w_2, \quad v_{2(m-1)!+1} \sim v_{3(m-1)!} \rightarrow w_3, \quad v_{3(m-1)!+1} \sim v_{4(m-1)!} \rightarrow w_4.$$ 

All other vertices correspond to $w_2$. Sets of one-dimensional simplices included in $S^1_i$ which are one-dimensional cycles are only the following $z$ and its counterpart $-z$:

$$z = \langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \cdots + \langle v_{m-1}, v_m \rangle + \langle v_m, v_1 \rangle.$$ 

Since $S^1_i$ does not have a two-dimensional simplex, $z$ is a representative element of homology classes of $S^1_i$. $z$ is transferred by $(f \circ A)_*$ to the following $z'$:

$$z' = \langle w_2, w_3 \rangle + \langle w_3, w_1 \rangle + \langle w_1, w_2 \rangle.$$ 

This is a cycle of $S^1_j$. Therefore, we have

$$(f \circ A)_* \neq 0.$$ 

Now we show the following lemma.

**Lemma 3**

(1) If a binary social choice rule satisfies acyclicity, Pareto principle and the minimal liberalism described in Assumption 1, then we obtain

$$(f \circ i)_* = 0 \quad \text{for all } i.$$ 

(2) If a binary social choice rule satisfies transitivity, Pareto principle and IIA, then we obtain (4).
Proof

(1) First we show \( (f \circ i)_* = 0 \) for individual A and B. Consider the case of individual B. From Assumption 1 and Pareto principle, the correspondences from the preference of individual B to the social preference when the preference of every other individual (including individual A and B) is fixed at \( v \) is obtained as follows:
\[
v_1 \sim v_{(m-1)!} \rightarrow w_2, \quad v_{(m-1)!+1} \sim v_m! \rightarrow w_1 \text{ or } w_2.
\]
In this case \( x_3 \) cannot be the most preferred alternative for the society.

Sets of one-dimensional simplices included in \( S^1_1 \) for individual B (denoted by \( S^1_B \)) which are one-dimensional cycles are only the following \( z \) and its counterpart \(-z\):
\[
z = \langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \cdots + \langle v_{m-1}, v_m \rangle + \langle v_m, v_1 \rangle.
\]
Since \( S^1_B \) does not have a two-dimensional simplex, \( z \) is a representative element of homology classes of \( S^1_B \) which is transferred by \( (f \circ i)_B \), the following cycle \( z' \) is obtained:
\[
z' = \langle w_2, w_1 \rangle + \cdots + \langle w_1, w_2 \rangle = 0 \quad \text{or} \quad z' = \langle w_2, w_2 \rangle = 0.
\]
This is not a cycle. Therefore, we get \( (f \circ i)_B = 0 \). Similarly we can show \( (f \circ i)_A = 0 \).

Next we show \( (f \circ i)_* = 0 \) for any individual (denoted by \( i \)) other than A and B. From Assumption 1 and Pareto principle, the correspondences from the preference of individual \( i \) to the social preference when the preference of every other individual (including individual A and B) is fixed at \( v_1 \) are obtained as follows:
\[
v_1 \sim v_m! \rightarrow w_2.
\]
Because \( x_3 \) and \( x_4 \) cannot be the most preferred alternative for the society. Then, we obtain \( (f \circ i)_* = 0 \) for all \( i \) other than A and B.

(2) By Pareto principle when the preference of every individual other than \( i \) is fixed at \( v_1 \), the correspondences from the preference of individual \( i \) to the social preference from \( v_1 \) to \( v_{(m-1)!} \) are as follows:
\[
v_1 \sim v_{(m-1)!} \rightarrow w_2.
\]

From Lemma 2 the correspondence from \( v_{(m-1)!+1} \) to the social preference is as follows:
\[
v_{(m-1)!+1} \rightarrow w_2.
\]

Consider another profile at which the preference of individual \( i \) changes to (234 ⋯ m1). By Pareto principle and the weak monotonicity (about \( x_1 \) and \( x_2 \)) the society prefers \( x_1 \) to all other alternatives. Further the weak monotonicity implies that the society prefers \( x_1 \) to all other alternatives so long as the most preferred alternative for all individuals other than \( i \) is \( x_1 \) regardless of the preference of individual \( i \). Thus, we obtain the following correspondences:
\[
v_{(m-1)!+2} \sim v_m! \rightarrow w_2.
\]
Sets of one-dimensional simplices included in \( S^1_1 \) which are one-dimensional cycles are only the following \( z \) and its counterpart \(-z\):
\[
z = \langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \cdots + \langle v_{m-1}, v_m \rangle + \langle v_m, v_1 \rangle.
\]
Since \( S^1_1 \) does not have a two-dimensional simplex, \( z \) is a representative element of homology classes of \( S^1_1 \) which is transferred by \( (f \circ i)_* \) to the following cycle \( z' \):
\[
z' = \langle w_2, w_2 \rangle = 0.
\]
Therefore, we have \( (f \circ i)_* = 0 \) for all \( i \). \( \square \)

The conclusion of this lemma contradicts (2) and (3) for both LP and AR. Therefore, we have shown the following theorem.

\(^{5}(f \circ i)_A, (f \circ i)_* \) for individual A. In this case \( x_3 \) cannot be the most preferred alternative for the society.
Theorem 1

(1) There exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism.

(2) There exists no binary social choice rule which satisfies transitivity, Pareto principle and IIA, and has no dictator.

We call the property expressed in (4) the *non-surjectivity of individual inclusion mappings*. Then, the above two theorems are special cases of the following theorem:

**Theorem 2.** There exists no binary social choice rule which satisfies Pareto principle and the non-surjectivity of individual inclusion mappings.

From (3) Pareto principle implies the *surjectivity of the diagonal mapping* \( (f \circ \Lambda_1) \neq 0 \), for binary social choice rules. Thus, this theorem is rewritten as follows:

There exists no binary social choice rule which satisfies the surjectivity of the diagonal mapping and the non-surjectivity of individual inclusion mappings.

4. Concluding remarks

We have shown the topological equivalence of the Arrow impossibility theorem that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives, and has no dictator, and Amartya Sen’s liberal paradox that there exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism. And we have also shown that these two theorems are special cases of the theorem that there exists no binary social choice rule which satisfies Pareto principle and the non-surjectivity of individual inclusion mappings.

In Baryshnikov [3] he said, “the similarities between the two theories, the classical and topological ones, are somewhat more extended than one would expect. The details seem to fit too well to represent just an analogy. I would conjecture that the homological way of proving results in both theories is a ‘true’ one because of its uniformity and thus can lead to much deeper understanding of the structure of social choice. To understand this structure better we need a much more evolved collection of examples of unifying these two theories and I hope this can and will be done”. This paper is an attempt to provide such an example.

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