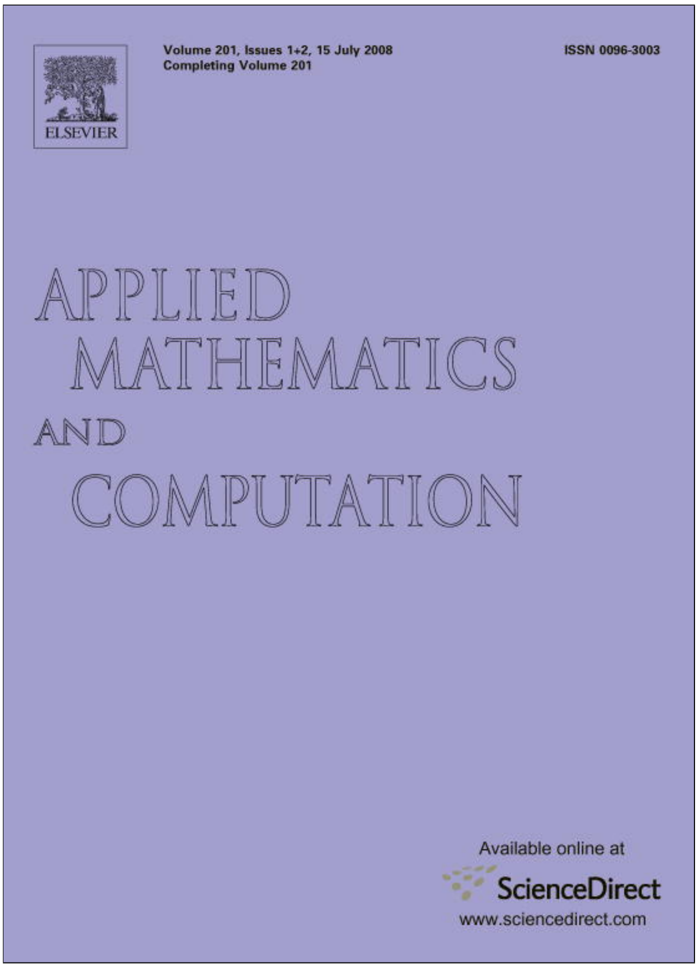
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Undecidability of Uzawa equivalence theorem and LLPO (Lesser limited principle of omniscience) ☆

Yasuhito Tanaka

Faculty of Economics, Doshisha University, Kamigyo-ku, Kyoto 602-8580, Japan

Abstract

The Uzawa equivalence theorem [H. Uzawa, Walras's Existence Theorem and Brouwer's Fixed Point Theorem, Economic Studies Quarterly 8 (1962) 59–62] showed (classically) that the existence of Walrasian equilibrium in an economy with continuous excess demand functions is equivalent to Brouwer's fixed point theorem, that is, the existence of a fixed point for any continuous function from an n-dimensional simplex to itself. We examine the Uzawa equivalence theorem from the point of view of constructive mathematics, and show that this theorem, properly speaking, the assumption of the existence of a Walrasian equilibrium price vector in this theorem, implies LLPO (Lesser limited principle of omniscience), and so it is non-constructive.

Keywords: Walrasian equilibrium; Uzawa equivalence theorem; LLPO (Lesser limited principle of omniscience)

1. Introduction

The existence of Walrasian equilibrium in an economy with continuous excess demand functions is proved by Brouwer's fixed point theorem. It is widely recognized that Brouwer's fixed point theorem is not a constructively proved theorem. The so-called Uzawa equivalence theorem [5] showed (classically) that the existence of a Walrasian equilibrium price vector is equivalent to Brouwer's fixed point theorem, that is, the existence of a fixed point for any continuous function from an n-dimensional simplex to itself. However, is this theorem constructively proved? In [6] Velupillai said that the Uzawa equivalence theorem implies decidability of the halting problem of the Turing machine. In this paper, we examine the Uzawa equivalence theorem from the point of view of *constructive mathematics*, and show that this theorem, properly speaking, the assumption of the existence of a Walrasian equilibrium price vector in this theorem, implies LLPO (Lesser limited principle of omniscience), and so it is non-constructive.

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E-mail address: yatanaka@mail.doshisha.ac.jp

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The omniscience principles are general statements that can be proved classically but not constructively, and are used to show that other statements do not admit constructive proofs.[1] This is done by showing that the statement implies an omniscience principle. The strongest omniscience principle is the law of excluded middle. A weaker one is the following limited principle of omniscience (abbreviated as LPO).

Limited principle of omniscience (LPO). Given a binary sequence $(a_n) = a_n, n \in \mathbb{N}$ (the set of positive integers), then either $a_n = 0$ for all n or $a_n = 1$ for some n.

Another omniscience principle is the following LLPO. It is weaker than LPO.

Lesser limited principle of omniscience (LLPO). Given a binary sequence (a_n) with at most one 1, then either $a_n = 0$ for all even n, or else $a_n = 0$ for all odd n.

In the next section, we present the theorem of the existence of Walrasian equilibrium and the Uzawa equivalence theorem with their classical proofs. In Section 3, we present some results of constructive mathematics, and prove that the assumption of the existence of Walrasian equilibrium in the Uzawa equivalence theorem implies LLPO.

2. Existence of Walrasian equilibrium and the Uzawa equivalence theorem

First we present the theorem of the existence of Walrasian equilibrium in an economy with continuous excess demand functions for the goods and its classical proof. Let Δ be an n-dimensional simplex ($n \geqslant 2$), and $p = (p_0, p_1, \ldots, p_n)$ be a point on Δ. $p_i \geqslant 0$ for each i and $\sum_{i=0}^n p_i = 1$. The prices of at least two goods are not zero. Thus, $p_i \neq 1$ for all i. Then, the theorem of the existence of Walrasian equilibrium is stated as follows.

Theorem 1 (Existence of Walrasian equilibrium). *Consider an economy with $n+1$ goods $X_0, X_1, \ldots, X_n$ with a price vector $p = (p_0, p_1, \ldots, p_n)$. Assume that an excess demand function for each good $f_i(p_0, p_1, \ldots, p_n), i = 0, 1, \ldots, n$, is continuous and satisfies the following condition*

$$p_0 f_0 + p_1 f_1 + \cdots + p_n f_n = 0 \quad \text{(Walras Law)}.$$

Then, there exists an equilibrium price vector $(p_0^*, p_1^*, \ldots, p_n^*)$ which satisfies $f_i(p_0, p_1, \ldots, p_n) \leqslant 0$ for all i $(i = 0, 1, \ldots, n)$. And when $p_i > 0$ we have $f_i(p_0^*, p_1^*, \ldots, p_n^*) = 0$.

Classical proof. See Appendix A. □

Next, we present the Uzawa equivalence theorem [5] which states that the existence of Walrasian equilibrium is equivalent to Brouwer's fixed point theorem, that is, the existence of a fixed point for any continuous function from an n-dimensional simplex to itself, and its classical proof.

Theorem 2 (Uzawa equivalence theorem). *The existence of Walrasian equilibrium is equivalent to Brouwer's fixed point theorem.*

Classical proof. We will show the converse of the previous theorem. Let $\psi = \{\psi_0, \psi_1, \ldots, \psi_n\}$ be an arbitrary continuous function from Δ to Δ, and construct excess demand functions by

$$z_i(p) = \psi_i(p) - p_i \mu(p), \ i = 0, 1, \ldots, n, \tag{1}$$

where $p = \{p_0, p_1, \ldots, p_n\}$, and $\mu(p)$ is defined as follows:

$$\mu(p) = \frac{\sum_{i=0}^n p_i \psi_i(p)}{\sum_{i=0}^n p_i^2}.$$

z_i for $i = 0, 1, \ldots, n$ are continuous, and as we will show below, they satisfy the Walras Law. Let multiply p_i to each z_i in (1), and summing up them from 0 to n, we obtain

$$\sum_{i=0}^n p_i z_i = \sum_{i=0}^n p_i \psi_i(p) - \mu(p) \sum_{i=0}^n p_i^2 = \sum_{i=0}^n p_i \psi_i(p) - \frac{\sum_{i=0}^n p_i \psi_i(p)}{\sum_{i=0}^n p_i^2} \sum_{i=0}^n p_i^2 = \sum_{i=0}^n p_i \psi_i(p) - \sum_{i=0}^n p_i \psi_i(p) = 0.$$

[1] About omniscience principles we refer to [1–4].

Thus, z_i for all i satisfy the conditions of excess demand functions, and by Theorem 1 there exists an equilibrium price vector. Let $p^* = \{p_0^*, p_1^*, \dots, p_n^*\}$ be an equilibrium price vector. Then we have

$$\psi_i(p^*) \leqslant \mu(p^*)p_i^*, \tag{2}$$

and if $p_i^* \neq 0$, $\psi_i(p^*) = \mu(p^*)p_i^*$. But since $\psi_i(p^*)$ must be nonnegative by its definition (a function from Δ to Δ), we have $\psi_i(p^*) = 0$ when $p_i^* = 0$. Therefore, for all i we obtain $\psi_i(p^*) = \mu(p^*)p_i^*$. Summing up them from $i = 0$ to n, we get

$$\sum_{i=0}^{n} \psi_i(p^*) = \mu(p^*) \sum_{i=0}^{n} p_i^*.$$

Because $\sum_{i=0}^{n} \psi_i(p^*) = 1,\ \sum_{i=0}^{n} p_i^* = 1$, we have $\mu(p^*) = 1$, and so we obtain

$$\psi_i(p^*) = p_i^*.$$

p^* is a fixed point of ψ. We have shown that any continuous function from Δ to Δ must have a fixed point. □

3. Uzawa equivalence theorem and LLPO

3.1. Basics of constructive mathematics

About major methods and principal results of constructive mathematics we refer to [1–4]. A real number is represented by rational approximations, and is identified with a sequence $x = (x_n)$ of rational numbers that is *regular* in the sense that

$$|x_m - x_n| \leqslant \frac{1}{m} + \frac{1}{n}$$

for all positive integers m and n. Two real numbers x and y are equal if $|x_n - y_n| \leqslant \frac{2}{n}$ for all positive integer n. Some operations on $\mathbb{R}$ (the set of real numbers) are defined as follows:

(1) $(x \pm y)_n = x_{2n} \pm y_{2n}$,
(2) $|x|_n = |x_n|$,

where $(x \pm y)_n$ denotes the nth term of the real number $x + y$ (or $x - y$), and $|x| = \max(x, -x)$. A real number $x = (x_n)$ is positive ($x > 0$) if there exists n such that $x_n > \frac{1}{n}$, and it is nonnegative ($x \geqslant 0$) if $x_n > -\frac{1}{n}$ for all n. x is negative ($x < 0$) if $-x$ is positive, that is, there exists n such that $-x_n > \frac{1}{n}$, then $x_n < -\frac{1}{n}$. Similarly, x is non-positive ($x \leqslant 0$) if $-x$ is nonnegative, that is, $-x_n > -\frac{1}{n}$ for all n, then $x_n < \frac{1}{n}$ for all n. For two real numbers x and y we define $x > y$ to mean $x - y > 0$. We obtain the following properties of positive real numbers.

(1) If $x > 0$ and $y > 0$, then $x + y > 0$.
It is clear.
(2) If $x + y > 0$, then $x > 0$ or $y > 0$.
If $x + y > 0$, there is a positive integer n such that $x_{2n} + y_{2n} > \frac{1}{n} = \frac{1}{2n} + \frac{1}{2n}$. Then, we have $x_{2n} > \frac{1}{2n}$ or $y_{2n} > \frac{1}{2n}$. This means $x > 0$ or $y > 0$.

If $x - y > 0$, for any real number z we have $(x - z) + (z - y) > 0$. Then, $x - z > 0$ or $z - y > 0$.

We need the following results.

Lemma 1

(1) *For any real number x there exists a binary sequence (a_n) such that*
 (*i*) *$x \leqslant 0$ if and only if $a_n = 0$ for all n.*
 (*ii*) *$x > 0$ if and only if $a_n = 1$ for some n.*

Conversely, for any binary sequence (a_n) *there exists a real number satisfying these two conditions. Therefore, for a real number* x *the property that* $x \leqslant 0$ *or* $x > 0$ *is equivalent to LPO.*

(2) *For any real number* x *there exists a binary sequence* (a_n) *with at most one* 1 *such that*
  (*i*) $x \geqslant 0$ *if and only if* $a_n = 0$ *for all even n.*
  (*ii*) $x \leqslant 0$ *if and only if* $a_n = 0$ *for all odd n.*

Conversely, for any binary sequence (a_n) *with at most one* 1 *there exists a real number satisfying these two conditions. Therefore, for a real number* x *the property that* $x \leqslant 0$ *or* $x \geqslant 0$ *is equivalent to LLPO.*

Proof

(1) For each positive integer n we have $x < \frac{1}{n}$ or $x > 0$. Define $a_n = 0$ if $x < \frac{1}{n}$ and $a_n = 1$ if $x > 0$. This defines a binary sequence (a_n). If $a_n = 0$ for all n, we have $x < \frac{1}{n}$ for all n, and it follows that $x \leqslant 0$. If $x \leqslant 0$ we have $a_n = 0$ for all n. On the other hand, if $a_n = 1$ for some n, we have $x > 0$. If $x > 0$, there exists an integer n such that $x > \frac{1}{n}$, and then we must have $a_n = 1$ for some n. Conversely, given a binary sequence (a_n), define

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

It is clear that $x \leqslant 0$ if and only if $a_n = 0$ for all n since if $a_n = 1$ for some n, $x \geqslant \frac{1}{2^n}$. And we have $x > 0$ if and only if $a_n = 1$ for some n since if $a_n = 0$ for all n, we have $x = 0$. Thus, x satisfies two conditions in (1).

(2) From (1) of this lemma we can construct a binary sequence (b_n) such that $|x| \leqslant 0$ if and only if $b_n = 0$ for all n, and $|x| > 0$ if and only if $b_n = 1$ for some n. Construct a binary sequence (a_n) as follows. When $b_1 = 0$, define $a_1 = 0$. When $b_1 = 1$, we have $|x| > 0$, and either $x > 0$ or $x < 0$. If $x > 0$, define $a_1 = 1$ and $a_n = 0$ for all $n \geqslant 2$. If $x < 0$, define $a_1 = 0$, $a_2 = 1$ and $a_n = 0$ for all $n \geqslant 3$. Assume $b_1 = 0$. When $b_2 = 0$, define $a_2 = 0$. When $b_2 = 1$, we have either $x > 0$ or $x < 0$. If $x > 0$, define $a_2 = 0$, $a_3 = 1$ and $a_n = 0$ for all $n \geqslant 4$. If $x < 0$, define $a_2 = 1$ and $a_n = 0$ for all $n \geqslant 3$. We proceed inductively. If $a_n = 0$ for all even n, $|x| \leqslant 0$ or $x > 0$, and if $a_n = 0$ for all odd n, $|x| \leqslant 0$ or $x < 0$. If $|x| \leqslant 0$, $a_n = 0$ for all n. If $x > 0$, $a_n = 1$ for some odd n and $a_n = 0$ for all even n, and if $x < 0$, $a_n = 1$ for some even n and $a_n = 0$ for all odd n.

Conversely, given a binary sequence (a_n) with at most one 1, define

$$x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a_n}{2^n}.$$

Then, it is clear that $x \geqslant 0$ if and only if $a_n = 0$ for all even n since if $a_n = 1$ for some even n, $x = -\frac{1}{2^n}$. Similarly we have $x \leqslant 0$ if and only if $a_n = 0$ for all odd n since if $a_n = 1$ for some odd n, $x = \frac{1}{2^n}$. Thus, x satisfies two conditions in (2). □

3.2. Uzawa equivalence theorem and LLPO

For all i other than 0 ψ_i is assumed to be defined as follows:

$$\psi_i = \frac{\lambda_i(p_i)}{\sum_{j=0}^{n} \lambda_j(p_j)}.$$

And for $i = 0$ we assume

$$\psi_0 = \frac{\lambda_0(p_0)}{\sum_{j=0}^{n} \lambda_j(p_j)}.$$

Then we have

$$z_i(p) = \frac{\lambda_i(p_i)}{\sum_{j=0}^{n} \lambda_j(p_j)} - p_i \frac{\sum_{j=0}^{n} p_j \lambda_j(p_j)}{\sum_{j=0}^{n} p_j^2 \sum_{j=0}^{n} \lambda_j(p_j)} \quad \text{for all } i \neq 0,$$

and

$$z_0(p) = \frac{\lambda_0(p_0)}{\sum_{j=0}^{n}\lambda_j(p_j)} - p_0\frac{\sum_{j=0}^{n}p_j\lambda_j(p_j)}{\sum_{j=0}^{n}p_j^2\sum_{j=0}^{n}\lambda_j(p_j)}.$$

If $z_i = 0$ for all i including $i = 0$, then we obtain

$$p_0\lambda_i(p_i) = p_i\lambda_0(p_0) \quad \text{for all } i \neq 0. \tag{3}$$

Now specifically we assume

$$\lambda_i(p_i) = p_i + 1,\ i \neq 0, \tag{4}$$

and

$$\lambda_0(p_0) = \begin{cases} \frac{np_0}{1-p_0} + \frac{1}{4} + b, & \text{when } p_0 < \frac{1}{4}, \\ \frac{np_0}{1-p_0} + p_0 + b, & \text{when } \frac{1}{4} \leqslant p_0 \leqslant \frac{1}{2}, \\ \frac{np_0}{1-p_0} + \frac{1}{2} + b, & \text{when } \frac{1}{2} < p_0 < 1, \end{cases} \tag{5}$$

where b is a real number such that $b > -\frac{1}{4}$. From (3) and (4) we have

$$p_i(\lambda_0(p_0) - p_0) = p_0, \quad i \neq 0. \tag{6}$$

This implies that all $p_i,\ i \neq 0$, are equal. Since $\sum_{j=0}^{n}p_j = np_i + p_0 = 1$ we have

$$p_i = \frac{1 - p_0}{n}. \tag{7}$$

If $p_0 = 0$, we have $p_i = \frac{1}{n}$ for all $i \neq 0$. But, then since $\lambda_0(p_0) = \frac{1}{4} + b > 0$ it contradicts (6). Thus, $p_0 \neq 0$. From (6) and (7)

$$(1 - p_0)(\lambda_0(p_0) - p_0) = np_0. \tag{8}$$

Therefore, from (5) and (8) we obtain

$$\begin{cases} p_0 - \frac{1}{4} - b = 0, & \text{when } p_0 < \frac{1}{4}, \\ b = 0, & \text{when } \frac{1}{4} \leqslant p_0 \leqslant \frac{1}{2}, \\ p_0 - \frac{1}{2} - b = 0, & \text{when } p_0 > \frac{1}{2}. \end{cases} \tag{9}$$

These are the equilibrium conditions. The assumption of the existence of Walrasian equilibrium implies the existence of p_0 in $(0, 1)$ such that one of these conditions is satisfied. Which of the conditions is satisfied depends on the value of b.

Now we show the following main result of this paper.

Lemma 2. *The existence of an equilibrium price vector assumed in the Uzawa equivalence theorem implies LLPO.*

Proof. Let p_0^* be an equilibrium value of p_0. If $b < 0$, we have $p_0^* < \frac{1}{4}$. If $b = 0$, p_0^* is any value in $\left[\frac{1}{4}, \frac{1}{2}\right]$. On the other hand, if $b > 0$, we have $p_0^* > \frac{1}{2}$. About three real numbers p_0^*, $\frac{1}{4}$ and $\frac{1}{2}$ we have $p_0^* > \frac{1}{4}$ or $p_0^* < \frac{1}{2}$. If $p_0^* > \frac{1}{4}$, then b must satisfy $b \geqslant 0$. And if $p_0^* < \frac{1}{2}$, then b must satisfy $b \leqslant 0$. Therefore, in order to determine an equilibrium price p_0^* we must know whether $b \geqslant 0$ or $b \leqslant 0$. As proved in (2) of Lemma 1 it implies LLPO. □

4. Final remark

The Uzawa equivalence theorem in general equilibrium theory demonstrates that the existence of Walrasian equilibrium in an economy with continuous excess demand functions is equivalent to Brouwer's fixed point theorem. We have shown that the existence of equilibrium price vector assumed in the Uzawa equivalence theorem implies LLPO (Lesser limited principle of omniscience). Therefore, it is non-constructive.

Appendix A

Proof of Theorem 1. Let v_i be a function from $p = (p_0, p_1, \ldots, p_n)$ to $v = (v_0, v_1, \ldots, v_n)$ as follows:

$$v_i = p_i + f_i, \quad \text{when } f_i > 0,$$
$$v_i = p_i, \quad \text{when } f_i \leqslant 0.$$

We construct a function $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_n)$ from Δ to Δ as follows:

$$\varphi_i(p_0, p_1, \ldots, p_n) = \frac{1}{v_0 + v_1 + \cdots + v_n} v_i.$$

Since we have $\varphi_i \geqslant 0,\ i = 0, 1, \ldots, n$, and

$$\varphi_0 + \varphi_1 + \cdots + \varphi_n = 1,$$

$(\varphi_0, \varphi_1, \ldots, \varphi_n)$ is a point on Δ.

Since each f_i is continuous, each φ_i is also continuous. Thus, by Brouwer's fixed point theorem there exists $p^* = (p_0^*, p_1^*, \ldots, p_n^*)$ that satisfies

$$(\varphi_0(p_0^*, p_1^*, \ldots, p_n^*), \varphi_1(p_0^*, p_1^*, \ldots, p_n^*), \ldots, \varphi_n(p_0^*, p_1^*, \ldots, p_n^*)) = (p_0^*, p_1^*, \ldots, p_n^*).$$

Since $v_i \geqslant p_i$ for all i, we have $v_i(p_0^*, p_1^*, \ldots, p_n^*) = \lambda p_i^*$ for all i for some $\lambda \geqslant 1$. We will show $\lambda = 1$. Now assume $\lambda > 1$. Then, if $p_i^* > 0$ we have $v_i(p_0^*, p_1^*, \ldots, p_n^*) > p_i^*$, that is, $f_i(p_0^*, p_1^*, \ldots, p_n^*) > 0$. On the other hand, since for all i $p_i^* \geqslant 0$ and the sum of them is one, at least one of them is positive. Then, we have $p_0^* f_0 + p_1^* f_1 + \cdots + p_n^* f_n > 0$. It contradicts the Walras Law. Therefore, we get $\lambda = 1$. And we obtain $v_0 = p_0^*, v_1 = p_1^*, \ldots, v_n = p_n^*$ and $f_i(p_0^*, p_1^*, \ldots, p_n^*) \leqslant 0$ for all i. □

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