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Stochastically stable states in an oligopoly with differentiated goods: equivalence of price and quantity strategies

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Abstract

This paper presents results on a *stochastically stable state* (or long-run equilibrium) in evolutionary dynamics based on imitation of strategies by firms in a symmetric oligopoly with differentiated goods. We consider both quantity setting and price-setting oligopoly games, and define a *globally surviving strategy (GSS)*. It is a strategy that can invade any configuration where the population monomorphically adopts some other strategy, and closely related to a *finite population evolutionarily stable strategy (ESS)*, defined by Schaffer (Schaffer, M.E., Evolutionarily stable strategies for a finite population and a variable contest size, Journal of Theoretical Biology 132, 1988, 469–478). In a quantity-setting oligopoly, the unique GSS output is equal to the unique finite population ESS output, and is a stochastically stable strategy. In a price-setting oligopoly, the unique GSS price is equal to the unique finite population ESS price, and is a stochastically stable strategy. The GSS in a quantity-setting oligopoly and that in a price-setting oligopoly are equivalent.

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Therefore, stochastically stable states in both cases coincide.

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1. Introduction

Oligopolistic markets have been analyzed based on one of two alternative assumptions about the behavior of firms, the quantity setting or Cournot approach and the price setting or Bertrand approach. It is well understood that in an oligopoly with substitutes, the Bertrand equilibrium is more efficient than the Cournot equilibrium (See Singh and Vives, 1984; Cheng, 1985; Vives, 1985). These analyses are based on the Nash equilibrium concept. In this paper, we present an evolutionary game theoretic analysis of oligopoly.

Vega-Redondo (1997) studied a stochastically stable state in a symmetric oligopoly with a homogeneous good, and showed that the Walrasian behavior, i.e., profit maximization given the market clearing price is a stochastically stable strategy. A stochastically stable state is a state where it spends most of the time in the *long run* when the probability of mutation becomes very small. Tanaka (1999) extended his result to a case of asymmetric homogeneous oligopoly with low cost and high cost firms, and showed that, under the assumption that the marginal cost is increasing, a stochastically stable output is equal to the competitive (Walrasian) output in each group of firms. Rhode and Stegeman (1998) analyzed Darwinian dynamics of a symmetric, differentiated duopoly with linear demand functions. They showed that firms' strategy choices cluster around a strategy profile that is not a one-shot Nash equilibrium, and this profile is invariant under a class of transformations of the strategy spaces (Bertrand vs. Cournot). They considered a stationary distribution of a Markov chain with large and frequent mutations. On the other hand, we consider the limit of a stationary distribution of a Markov chain as mutations vanish according to the formulation of evolutionary dynamics used by Robson and Vega-Redondo (1996) and Vega-Redondo (1997).

Schaffer (1988) proposed a concept of an evolutionarily stable strategy (ESS) for a finite (or small) population as a generalization of the standard ESS concept for an infinite (or large) population by Maynard Smith (1982). In this paper, it is called a *finite population ESS*. He showed that a finite population ESS is not generally a Nash equilibrium strategy. In Schaffer (1989), he applied this concept to an economic game, and showed that the strategy that survives in economic natural selection is the relative, not absolute, payoff maximizing strategy. He considered the following survival rule in economic natural selection. Firms are born with strategies and cannot change their strategies in response to changing circumstances. At the end of each period, if the payoff of Firm i is larger than the payoff of Firm j, the probability that Firm i survives in the next period is larger than the probability that Firm j survives in the next period. Alternatively, we can consider that the survival rule operates on strategies, not firms, and the proportion of successful strategies in the population grows by firms' imitation of strategies. [1]

In this paper, we define a *globally surviving strategy (GSS)*. It is a strategy that, if adopted by a single mutant, can invade any configuration where the population monomorphically adopts some other strategy, and closely related to Schaffer's finite population ESS. And we present evolutionary game theoretic analyses of symmetric price setting and quantity-setting oligopolies with differentiated goods. Firms can observe decisions of other firms and realizations of variables (outputs or prices) other than their strategic variables, but do not know demand functions. Hence, they cannot compute the best responses to the strategies of other firms, but can only imitate the strategies of the firms, which are currently making the highest profit. On the other hand, firms know that the demand functions for all firms are symmetric, and their cost functions are the same. If all firms choose the same strategy (output or price), denoting it by s_1, in a symmetric oligopoly their profits are equal, then there is nobody to imitate. Now, suppose that one firm experiments with a different strategy, s_2. If this firm makes higher profit than the rest of the firms, they will wish to imitate its success, and then s_1 is unstable. If, starting from any strategy other than s_2, experimenting with s_2 always leads to higher profit for the experimenter than for non-experimenting firms, then all firms playing a strategy other than s_2 will wish to imitate s_2. This is precisely what is required for s_2 to be a GSS. On the other hand, if the firm experimenting with s_2 makes lower profit than the rest of the firms, then they will not wish to imitate its failure; and, in fact, the experimenter will wish to switch from s_2 back to s_1. If, starting from s_1, experimenting always leads to lower profit for the experimenter than for non-experimenting firms, then there is no strategy that firms playing s_1 will wish to imitate, and s_1 is stable. This is precisely what is required for s_1 to be a finite population ESS. We will show that the first-order condition for the GSS and that for the finite population ESS are the same, but the second-order conditions are different in each of quantity setting and price-setting oligopolies.

[1] Hansen and Samuelson (1988) also presented analyses about evolution in economic games. They showed that with small number of firms the surviving strategy in economic natural selection, they called such a strategy a *universal survival strategy*, is not a Nash equilibrium strategy. Their universal survival strategy is essentially equivalent to Schaffer's finite population ESS. They said, ''In real-world competition, firms will be uncertain about the profit outcomes of alternative strategies. This presents an obvious obstacle to instantaneous optimization. Instead, firms must search for and learn about more profitable strategies. As Alchian (1950) emphasizes, an important mechanism for such a search depends on a comparison of observed profitability across the strategies used by market participants. That is, search for better strategies is based on relative profit comparisons''. About more recent analyses of an imitation behavior, see Schlag (1998, 1999).

The mechanism of imitative evolutionary dynamics will be explained in Section 3. Some recent papers, such as Robson and Vega-Redondo (1996) and Vega-Redondo (1997), considered a model of stochastic evolutionary dynamics assuming imitation dynamics of players' strategies. On the other hand, other some papers, such as Kandori and Rob (1995, 1988) and Galesloot and Goyal (1997), considered best-response dynamics. In best-response dynamics, each player chooses a strategy in period $t+1$, which is a best response to other players' strategies in period t. Thus, players must know the structure of the game, and must be able to compute the best responses. In imitation dynamics, players simply mimic successful other players' strategies. We think that imitation dynamics is more appropriate than best-response dynamics for an economic game with boundedly rational players.

We are concerned with showing the following results. If the goods of the firms are substitutes, in a quantity-setting oligopoly, the unique GSS output is between the Nash–Cournot equilibrium output and the competitive output (at which the price is equal to the marginal cost); and in a price-setting oligopoly, the unique GSS price is between the Nash–Bertrand equilibrium price and the competitive price (which is equal to the marginal cost). If the goods of the firms are complements, the GSS output is smaller than the Nash equilibrium and competitive outputs, and the GSS price is higher than the Nash equilibrium and competitive prices. The GSS in a quantity-setting oligopoly and that in a price-setting oligopoly are equivalent. The state in which all firms choose the GSS is a stochastically stable state in each case. Therefore, the stochastically stable state in a quantity-setting oligopoly and that in a price-setting oligopoly coincide.

In the next section, we consider globally surviving strategies (GSSs) in quantity setting and price-setting oligopolies, and show the equivalence of the GSSs in both cases. Also we will show that the GSS is the same as the finite population ESS in each case, but with different second-order conditions. In Section 3, we will show that the GSS, which is also the finite population ESS, is a stochastically stable strategy in each case. From the equivalence of the GSSs we get a conclusion that the stochastically stable state in a quantity-setting oligopoly and that in a price-setting oligopoly coincide. In Section 4, we present an interpretation of the results.

2. Globally surviving strategies and finite population ESSs

2.1. The model

Consider an oligopoly in which there are $n(\geqq 2)$ firms; n is an integer number. The firms produce differentiated goods. The output of Firm i, $i=1, 2, \ldots, n$, is denoted by x_i and the price of the good of Firm i is denoted by p_i.

The (ordinary) demand functions for the goods are:

$$x_i = x_i(\boldsymbol{p}), \qquad i = 1, 2, \ldots, n, \qquad \text{where } \boldsymbol{p} = \{p_1, p_2, \ldots, p_n\}. \tag{1}$$

Inverting the demand functions in Eq. (1), we obtain the following inverse demand functions:

$$p_i = p_i(\boldsymbol{x}), \qquad i = 1, 2, \ldots, n, \qquad \text{where } \boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}. \tag{2}$$

About $x_i(\boldsymbol{p})$ and $p_i(\boldsymbol{x})$, we assume the following regular assumptions.

Assumption 1. $x_i(\boldsymbol{p})$ and $p_i(\boldsymbol{x})$ are symmetric for all firms, twice continuously differentiable, and satisfy the following relations:

$$\frac{\partial x_i}{\partial p_i} < 0, \text{and} \left|\frac{\partial x_i}{\partial p_i}\right| > \left|\frac{\partial x_j}{\partial p_i}\right|, \text{for all } i, \text{and } j \neq i,$$

and:

$$\frac{\partial p_i}{\partial x_i} < 0, \text{and} \left|\frac{\partial p_i}{\partial x_i}\right| > \left|\frac{\partial p_j}{\partial x_i}\right|, \text{for all } i, \text{and } j \neq i.$$

If $(\partial x_j/\partial p_i) > 0$ and $(\partial p_j/\partial x_i) < 0$, the goods of the firms are substitutes, and if $(\partial x_j/\partial p_i) < 0$ and $(\partial p_j/\partial x_i) > 0$, each two goods are complements.

The common marginal cost for the firms is c. The fixed cost is zero.

In this section, we assume that all variables are continuous.

2.2. Globally surviving strategies

In this subsection, we consider globally surviving strategies.

In a quantity-setting oligopoly the profits of the firms are:

$$\pi_i(\boldsymbol{x}) = p_i(\boldsymbol{x})x_i - cx_i, \qquad i = 1, 2, \ldots, n. \tag{3}$$

We consider an evolutionary game in which n firms repeatedly play an n firms oligopoly stage game. In this evolutionary game, the population is n, and the stage game is also an n-player game. Thus, it is a so-called *playing-the-field model*. Strategies for the firms are their outputs. The firms repeatedly play the stage game in each period, and may change their strategies between one period and the next period.

Consider a state in which all firms choose a strategy x'. If, when only one firm (an experimenter or a mutant firm) chooses a different strategy x^*, its profit is larger than the profits of the rest of the firms, and this relation holds for all $x' \neq x^*$, then we define x^* as a *GSS*. Without loss of generality, assuming that the mutant firm is Firm j, x^* is a GSS if:

$$\pi_j(\boldsymbol{x}) > \pi_i(\boldsymbol{x}) \text{ for all } i \neq j \text{ and all } x' \neq x^*, \tag{4}$$

where:

$$x_j = x^*, \text{ and } x_i = x' \neq x^* \text{ for all } i \neq j \text{ in } \boldsymbol{x}.$$

Firm i is a representative non-mutant firm. The GSS is a strategy that, if adopted by a single mutant, can invade any configuration, where the population monomorphically adopts some other strategy.

We define:

$$\varphi(x', x_j) = \pi_i(\boldsymbol{x}) - \pi_j(\boldsymbol{x}),$$

where:

$$x_i(=x') \text{ are equal for all } i \neq j \text{ in } \boldsymbol{x}.$$

Consider the following maximization problem:

$$\max_{x'} \varphi(x', x_j) \text{ given } x_j. \tag{5}$$

A necessary and sufficient condition for x^* to be a unique GSS is that it is the unique solution of Eq. (5) when that solution is equal to x_j, i.e.:

$$x^* = \arg\max_{x'} \varphi(x', x^*).$$

The reason is as follows. If x^* is the unique solution of Eq. (5), given $x_j = x^*$, then $x_j = x^*$ satisfies Eq. (4) since φ has the maximum value, which is zero, only when $x_i = x^*$ for all $i \neq j$.

Substitution of Eq. (3) into φ, replacing x_j and x_i, respectively, by x^* and x', yields:

$$\varphi = p_i(\boldsymbol{x})x' - cx' - p_j(\boldsymbol{x})x^* + c^*.$$

The first-order condition for maximization of φ is obtained by differentiating φ with respect to x', given that $x_i = x'$ for all $i \neq j$, as follows:

$$p_i - c + x'\left[\frac{\partial p_i}{\partial x_i} + (n-2)\frac{\partial p_i}{\partial x_k}\right] - x^*\left[\frac{\partial p_j}{\partial x_i} + (n-2)\frac{\partial p_j}{\partial x_k}\right] = 0,$$

$$\text{where } k \neq i, j. \tag{6}$$

In symmetric situations where all x's are equal, we have $\partial p_i/\partial x_k = \partial p_j/\partial x_k = \partial p_j/\partial x_i$. Thus, the condition for the GSS is rewritten as follows:

$$p_i - c + x^* \left(\frac{\partial p_i}{\partial x_i} - \frac{\partial p_j}{\partial x_i} \right) = 0. \tag{7}$$

It can be thought of as a first-order condition for a firm that maximizes the difference between its own profit and that of its competitor. We call x^* derived from Eq. (7) the *GSS output*.

The second-order condition for the GSS is obtained by differentiating Eq. (6) with respect to x', given that $x_i = x'$ for all $i \neq j$, as follows:

$$2\left[\frac{\partial p_i}{\partial x_i} + (n-2)\frac{\partial p_i}{\partial x_k} \right] + x'\left[\frac{\partial^2 p_i}{\partial x_i^2} + 2(n-2)\frac{\partial^2 p_i}{\partial x_i x_k} + (n-2)^2 \frac{\partial^2 p_i}{\partial x_k^2} \right]$$
$$- x^*\left[\frac{\partial^2 p_j}{\partial x_i^2} + 2(n-2)\frac{\partial^2 p_j}{\partial x_i x_k} + (n-2)^2 \frac{\partial^2 p_j}{\partial x_k^2} \right] < 0,$$
$$\text{where } k \neq i, j. \tag{8}$$

We assume that Eq. (8) holds for all values of x_i (with $x_k = x_i$) not just when $x_i = x_j$. Then, the GSS output is unique.

Now we will show:

Proposition 1. *(1) If the goods of the firms are substitutes, the GSS output in a quantity-setting oligopoly is between the Nash–Cournot equilibrium output and the competitive (or Walrasian) output.*
(2) If the goods are complements, the GSS output in a quantity-setting oligopoly is smaller than both the Nash–Cournot equilibrium output and the competitive output.

PROOF. The condition for the Nash–Cournot equilibrium output, x_c, is:

$$\frac{\partial \pi_i}{\partial x_i} = p_i - c + x_c \frac{\partial p_i}{\partial x_i} = 0.$$

On the other hand, the condition for the competitive output, x_w, is:

$$p_i - c = 0.$$

When $x_i = x_c$ for all i, the left-hand side of Eq. (7) is equal to $-x_c(\partial p_j/\partial x_i)$ and is strictly positive. On the other hand, when $x_i = x_w$ for all i, the

left-hand side of Eq. (7) is equal to $x_w[(\partial p_i/\partial x_i) - (\partial p_j/\partial x_i)]$ and is strictly negative. Thus, x^* is between x_c and x_w. Since x_c is usually smaller than x_w, we have $x_c < x^* < x_w$.
(2) If the goods are complements, when $x_i = x_c$ for all i, the left-hand side of Eq. (7) is strictly negative. So we have $x^* < x_c < x_w$. (Q.E.D.)

In a price-setting oligopoly, the profits of the firms are:

$$\pi_i(\boldsymbol{p}) = p_i x_i(\boldsymbol{p}) - c x_i(\boldsymbol{p}), \qquad i = 1, 2, \dots, n.$$

We consider an evolutionary game in which n firms repeatedly play an oligopoly stage game, and define the GSS price similarly to the quantity-setting oligopoly. Strategies for the firms in this game are the prices of their goods. Denote the GSS price by p^*.

p^* is obtained as the unique solution of the following maximization problem:

$$p^* = \arg\max_{p'} \left[p_i x_i(\boldsymbol{p}) - c x_i(\boldsymbol{p}) - p_j x_j(\boldsymbol{p}) + c x_j(\boldsymbol{p}) \right],$$

where:

$$p_j = p^*, \text{and } p_i = p' \text{ for all } i \neq j \text{ in } \boldsymbol{p}.$$

The first-order condition for the GSS price is:

$$x_i + (p' - c)\left[\frac{\partial x_i}{\partial p_i} + (n-2)\frac{\partial x_i}{\partial p_k}\right] - (p^* - c)\left[\frac{\partial x_j}{\partial p_i} + (n-2)\frac{\partial x_j}{\partial p_k}\right] = 0,$$

where $k \neq i, j$.

In symmetric situations, this is rewritten as follows:

$$x_i + (p^* - c)\left(\frac{\partial x_i}{\partial p_i} - \frac{\partial x_j}{\partial p_i}\right) = 0. \tag{9}$$

The second-order condition for the GSS price is:

$$2\left[\frac{\partial x_i}{\partial p_i} + (n-2)\frac{\partial x_i}{\partial p_k}\right] + (p' - c)\left[\frac{\partial^2 x_i}{\partial p_i^2} + 2(n-2)\frac{\partial^2 x_i}{\partial p_i p_k} + (n-2)^2\frac{\partial^2 x_i}{\partial p_k^2}\right] - (p^* - c)\left[\frac{\partial^2 x_j}{\partial p_i^2} + 2(n-2)\frac{\partial^2 x_j}{\partial p_i p_k} + (n-2)^2\frac{\partial^2 x_j}{\partial p_k^2}\right] < 0, \qquad \text{where } k \neq i, j. \tag{10}$$

We assume that Eq. (10) holds for all values of p_i (with $p_k = x_i$) not just when $p_i = p_j$. Then, the GSS price is unique.

Now we can show:

Proposition 2. *(1) If the goods of the firms are substitutes, the GSS price in a price-setting oligopoly is between the Nash–Bertrand equilibrium price and the competitive (or Walrasian) price.*
(2) If the goods are complements, the GSS price in a price-setting oligopoly is higher than both the Nash–Bertrand equilibrium price and the competitive price.

PROOF. (1) The condition for the Nash–Bertrand equilibrium price, p_b, is:

$$\frac{\partial \pi_i}{\partial p_i} = x_i + (p_b - c)\frac{\partial x_i}{\partial p_i} = 0. \tag{11}$$

Since $(\partial x_i/\partial p_i) < 0$, we have $p_b > c$. The competitive price, p_w, is equal to the marginal cost, and we have $p_w = c$. When $p_i = p_b$ for all i, the left-hand side of Eq. (9) is equal to $-(p_b - c)(\partial x_j/\partial p_i)$ and is strictly negative. On the other hand, when $p_i = p_w$ for all i, the left-hand side of Eq. (9) is equal to x_i and is strictly positive. Thus, p^* is between p_b and p_w. Since p_b is usually higher than p_w, we have $p_b > p^* > p_w$.
(2) If the goods are complements, when $p_i = p_b$ for all i, the left-hand side of Eq. (9) is strictly positive. So we have $p^* > p_b > p_w$. (Q.E.D.)

2.3. The equivalence of dual GSSs

The inverse demand functions in a quantity-setting oligopoly given in Eq. (2) have been obtained by inverting the demand functions in a price-setting oligopoly given in Eq. (1). These two systems represent the same demand structure.

Considering the effects of a change in p_i on x_i for all i in Eq. (2), keeping p_k for $k \neq i$ constant, yields:

$$1 = \frac{\partial p_i}{\partial x_i}\frac{\partial x_i}{\partial p_i} + \sum_{j=1,\, j\neq i}^{n} \frac{\partial p_i}{\partial x_j}\frac{\partial x_j}{\partial p_i}, \tag{12}$$

and:

$$0 = \frac{\partial p_k}{\partial x_k}\frac{\partial x_k}{\partial p_i} + \sum_{j=1,\, j\neq k}^{n} \frac{\partial p_k}{\partial x_j}\frac{\partial x_j}{\partial p_i} \text{ for all } k \neq i. \tag{13}$$

In symmetric situations in which all x_i's are equal, we have:

$$\frac{\partial p_k}{\partial x_k} = \frac{\partial p_i}{\partial x_i} \text{ for all } k \neq i,$$

and:

$$\frac{\partial p_i}{\partial x_k} = \frac{\partial p_j}{\partial x_i} \text{ for all } j \neq i \text{ and } l \neq k.$$

Denote them, respectively, by $\partial p_i / \partial x_i$ and $\partial p_j / \partial x_i$. Also, from the symmetry, $\partial x_j / \partial p_i$ for all $j \neq i$ are equal. Then, from Eqs. (12) and (13), we obtain:

$$1 = \frac{\partial p_i}{\partial x_i}\frac{\partial x_i}{\partial p_i} + (n-1)\frac{\partial p_j}{\partial x_i}\frac{\partial x_j}{\partial p_i}, \qquad \text{where } j \neq i, \tag{14}$$

and:

$$0 = \frac{\partial p_i}{\partial x_i}\frac{\partial x_k}{\partial p_i} + \frac{\partial p_j}{\partial x_i}\frac{\partial x_i}{\partial p_i} + (n-2)\frac{\partial p_j}{\partial x_i}\frac{\partial x_j}{\partial p_i}, \tag{15}$$

$$= \frac{\partial p_j}{\partial x_i}\frac{\partial x_i}{\partial p_i} + \left[\frac{\partial p_i}{\partial x_i} + (n-2)\frac{\partial p_j}{\partial x_i}\right]\frac{\partial x_j}{\partial p_i} \text{ for all } k \neq i, \qquad \text{where } j \neq i.$$

We use $\partial x_k / \partial p_i = \partial x_j / \partial p_i$ in Eq. (15). From Eqs. (14) and (15), we obtain:

$$\frac{\partial x_i}{\partial p_i} = \frac{1}{D}\left[\frac{\partial p_i}{\partial x_i} + (n-2)\frac{\partial p_j}{\partial x_i}\right], \tag{16}$$

and:

$$\frac{\partial x_j}{\partial p_i} = -\frac{1}{D}\frac{\partial p_j}{\partial x_i}, \tag{17}$$

where:

$$D = \left(\frac{\partial p_i}{\partial x_i} - \frac{\partial p_j}{\partial x_i}\right)\left[\frac{\partial p_i}{\partial x_i} + (n-1)\frac{\partial p_j}{\partial x_i}\right].$$

Substituting Eqs. (16) and (17) into Eq. (9), since $[(\partial p_i / \partial x_i) - (\partial p_j / \partial x_i)] < 0$, we find:

$$p^* - c + \left(\frac{\partial p_i}{\partial x_i} - \frac{\partial p_j}{\partial x_i}\right)x_i = 0.$$

This is equivalent to Eq. (7), which is the condition for the GSS in a quantity-setting oligopoly. Therefore, we obtain:

Theorem 1. *The GSS in a quantity-setting oligopoly and that in a price-setting oligopoly are equivalent.*

This means that the GSS output in a quantity-setting oligopoly and the output with the GSS price in a price-setting oligopoly are equal, and the GSS price in a price-setting oligopoly and the price with the GSS output in a quantity-setting oligopoly are equal.

2.4. Finite population ESSs

In this subsection, we consider finite population ESSs.

Consider a state in which all firms choose a strategy $\bar{x}^*$ If, when one firm (an experimenter or a mutant firm) chooses a different strategy x', its profit is smaller than the profits of the rest of the firms, and this relation holds for all $x' \neq \bar{x}^*$, then $\bar{x}^*$ is a *finite population ESS.*[2] Assuming that the mutant firm is Firm i, $\bar{x}^*$ is a finite population ESS if:

$$\pi_j(\boldsymbol{x}) > \pi_i(\boldsymbol{x}) \text{ for all } j \neq i \text{ and all } x' \neq \bar{x}^*, \tag{18}$$

where:

$$x_i = x' \neq \bar{x}^*, \text{and } x_j = \bar{x}^* \text{ for all } j \neq i \text{ in } \boldsymbol{x}.$$

We call $\bar{x}^*$ the *finite population ESS output*. We define:

$$\psi(x', x_j) = \pi_j(\boldsymbol{x}) - \pi_i(\boldsymbol{x}),$$

where:

$$x_i = x', \text{and } x_j \text{ are equal for all } j \neq i \text{ in } \boldsymbol{x}.$$

Consider the following maximization problem:

$$\max_{x'} \psi(x', x_j) \text{ given } x_j \text{ for all } j \neq i. \tag{19}$$

A necessary and sufficient condition for $\bar{x}^*$ to be a finite population ESS is that it is the unique solution of Eq. (19) when that solution is equal to x_j, i.e.:

$$\bar{x}^* = \arg\max_{x'} \psi(x', \bar{x}^*).$$

The reason is as follows. If $\bar{x}^*$ is the unique solution of Eq. (19), given $x_j = \bar{x}^*$ for all $j \neq i$, then $x_i = \bar{x}^*$ satisfies Eq. (18) since ψ has the maximum value, which is zero, only when $x_i = \bar{x}^*$.

[2] Schaffer's original definition is weaker. He defines $\bar{x}^*$ as a finite population ESS if Eq. (18) is satisfied with weak inequality. We adopt the definition with strong inequality. About the definition of a finite population ESS, see Crawford (1991).

Substitution of Eq. (3) into ψ, replacing x_j and x_i, respectively, by $\bar{x}^*$ and x', yields:

$$\psi = p_i(\boldsymbol{x})\,x' - cx' - p_j(\boldsymbol{x})\,\bar{x}^* + c\bar{x}^*.$$

The first-order condition for maximization of ψ with respect to x' is:

$$p_i - c + x'\frac{\partial p_i}{\partial x_i} - \bar{x}^*\frac{\partial p_j}{\partial x_i} = 0.$$

From symmetry of the oligopoly, this is rewritten as follows:

$$p_i - c + \bar{x}^*\left(\frac{\partial p_i}{\partial x_i} - \frac{\partial p_j}{\partial x_i}\right) = 0.$$

This is the same as Eq. (7), which is the first-order condition for the GSS output.

The second-order condition for the finite population ESS output is:

$$2\frac{\partial p_i}{\partial x_i} + x'\frac{\partial^2 p_i}{\partial x_i^2} - \bar{x}^*\frac{\partial^2 p_j}{\partial x_i^2} < 0. \tag{20}$$

We assume that Eq. (20) holds for all values of x_i. Then, the finite population ESS output is unique.

Eq. (20) is different from the second-order condition for the GSS output given in Eq. (8). Since the first-order condition for the GSS output and that for the finite population ESS output are the same, the unique GSS output and the unique finite population ESS output are equal so long as they exist. But since the second-order conditions are different, the condition for the existence of a unique GSS output and that of a unique finite population ESS output are different.

In a price-setting oligopoly the finite population ESS price, p^*, is obtained as the unique solution of the following problem:

$$\bar{p}^* = \arg\max_{p'}\left[\,p_i x_i(\boldsymbol{p}) - cx_i(\boldsymbol{p}) - p_j x_j(\boldsymbol{p}) + cx_j(\boldsymbol{p})\right],$$

where:

$$p_i = p', \text{and } p_j = \bar{p}^* \text{ for all } j \neq i \text{ in } \boldsymbol{p}.$$

The first-order condition for the finite population ESS price is:

$$x_i + (p' - c)\frac{\partial x_i}{\partial p_i} - (\bar{p}^* - c)\frac{\partial x_j}{\partial p_i} = 0.$$

From symmetry of the oligopoly, this is rewritten as:

$$x_i + (\bar{p}^* - c)\left(\frac{\partial x_i}{\partial p_i} - \frac{\partial x_j}{\partial p_i}\right) = 0.$$

This is the same as Eq. (9), which is the first-order condition for the GSS price.

The second-order condition for the finite population ESS price is:

$$2\frac{\partial x_i}{\partial p_i} + (p' - c)\frac{\partial^2 x_i}{\partial p_i^2} - (\bar{p}^* - c)\frac{\partial^2 x_j}{\partial p_i^2} < 0. \qquad (21)$$

We assume that Eq. (21) holds for all values of p_i. Then, the finite population ESS price is unique.

Eq. (21) is different from the second-order condition for the GSS price given in Eq. (10). Since the first-order condition for the GSS price and that for the finite population ESS price are the same, the unique GSS price and the unique finite population ESS price are equal so long as they exist. But since the second-order conditions are different, the condition for the existence of a unique GSS price and that of a unique finite population ESS price are different.

We assume that there exist the unique GSS output and the unique finite population ESS output, and that there exist the unique GSS price and the unique finite population ESS price. In the following example, we explicitly obtain the GSS and ESS output and the price in a linear demand case.

Summarizing the results:

Proposition 3. *The GSS output and the finite population ESS output are equal, and the GSS price and the finite population ESS price are equal.*

2.5. An example

In this subsection, we consider an example with linear demand functions.

The inverse demand functions are assumed as follows:

$$p_i = a - x_i - b\sum_{j=1,\, j\neq i}^{n} x_j, \qquad i = 1, 2, \ldots, n, \qquad \text{where } |b| < \frac{1}{n-1}.$$

And we assume $a > c$. From Eq. (7):

$$a - [1 + (n-1)b]x^* - (1-b)x^* - c = 0.$$

Then, we obtain:

$$x^* = \frac{a-c}{2+(n-2)b}.$$

The second-order condition for the finite population ESS output (20) globally holds since $(\partial p_i/\partial x_i) = -1 < 0$. The second-order condition for the GSS output (8) also globally holds since $(\partial p_i/\partial x_i) + (n-2)(\partial p_i/\partial x_k) = -1-(n-2)b < 0$.

From the inverse demand functions, the (ordinary) demand functions are derived as follows:

$$x_i = \frac{1}{(1-b)[1+(n-1)b]}\left\{(1-b)a - [1+(n-2)b]p_i + b\sum_{j=1,\,j\neq i}^{n} p_j\right\}, \qquad i=1,2,\ldots,n.$$

From Eq. (9):

$$(1-b)a - [1+(n-2)b]p^* + b(n-1)p^* - (p^*-c)[1+(n-2)b+b] = 0.$$

Then, we obtain:

$$p^* = \frac{(1-b)a + [1+(n-1)b]c}{2+(n-2)b}.$$

The second-order condition for the finite population ESS price (21) globally holds since $(\partial x_i/\partial p_i) = -([1+(n-2)b]/\{(1-b)[1+(n-1)b]\}) < 0$. The second-order condition for the GSS price (10) also globally holds since:

$$\frac{\partial x_i}{\partial p_i} + (n-2)\frac{\partial x_i}{\partial p_k} = \frac{1}{(1-b)[1+(n-1)b]}\{-[1+(n-2)b] + (n-2)b\}, = -\frac{1}{(1-b)[1+(n-1)b]} < 0.$$

3. Stochastically stable states

In this section, we will show that the GSS output and price obtained in the previous section, which are also the finite population ESS output and price, are stochastically stable strategies in evolutionary dynamics.

Kandori and Rob (1995), Kandori et al. (1993), Robson and Vega-Redondo (1996) and Vega-Redondo (1997) presented analyses of stochastically stable states of dynamic and stochastic evolutionary games. In our model, n players (firms) play a symmetric oligopoly game in each period. According to Robson and Vega-Redondo (1996) and Vega-Redondo (1997), we consider the following imitation dynamics of the firms' strategies. In period $t+1$, each firm has a chance with positive probability less than one to change its strategy to the strategy that achieved the highest profit in period t among the strategies chosen by the firms in period t. If a firm chose a strategy that achieved the strictly highest profit in period t, this firm does not change the strategy. If in period t the highest profit were attained by two or more firms even if they chose different strategies, in period $t+1$, each firm may choose either strategy among the strategies that attained the highest profit in period t. If all firms chose the same strategy in period t, this strategy achieved the strictly highest profit, then no firm changes the strategy.

First consider the quantity-setting oligopoly. In this section, like in Vega-Redondo (1997), instead of assuming continuous variables, we assume that the firms must choose their outputs from a finite grid $\Gamma = \{0, \delta, 2\delta, \dots, \upsilon\delta\}$ where $\delta > 0$ and $\upsilon \in \mathbb{N}$ are arbitrary. It is required that the GSS (and the finite population ESS) output belongs to this grid. A state of the imitation dynamics is identified with an output profile. The state space is denoted by Ω, which is equal to Γ^n. Denote the transition matrix of this dynamics by $\mathbf{T}(\omega, \omega')$, and by $\mathbf{T}^{(m)}(\omega, \omega')$ the corresponding m-step transition matrix, where $\omega, \omega' \in \Omega$.

In addition to this dynamic adjustment, there is a random mutation. In each period, each firm switches (mutates) its strategy with probability ε. Mutation may be interpreted as experimentation of a new strategy by the firms. Any strategy may be chosen with positive probability. Thus, all elements of the transition matrix with mutations are strictly positive. Such a stochastic process is called an *Ergodic Markov chain*, and it has a unique stationary distribution. By the Ergodic Theorem, the stationary distribution represents the frequency distribution of states of the Markov chain over time. Consider the limit of the stationary distribution as $\varepsilon \to 0$. The stochastically stable states are the states that are assigned positive probability in the limit.[3]

[3] This adjustment process is the same as that in Robson and Vega-Redondo (1996) and Vega-Redondo (1997). It has the stochastic nature even without mutation since each firm has a chance to change its strategy independently with some positive probability, and the number of firms who change the strategies in period $t+1$ to the most profitable strategy in period t is a stochastic variable without mutation. In period $t+1$, all firms may choose the most profitable strategy in period t with strictly positive probability.

We define a *limit set* of the dynamics without mutation. A set of states $A \in \Omega$ is a limit set of **T** if this set is closed under the finite chains of positive probability transitions, i.e.:

$$(1)\ \forall \omega \in A, \forall \omega' \notin A, T(\omega, \omega') = 0.$$

$$(2)\ \forall \omega \in A,\ \omega' \in A, \exists m \in \mathbb{N} \text{ such that } T^{(m)}(\omega, \omega') > 0.$$

Any set that include a single state in which all firms choose the same strategy is a limit set since in such a state no firm changes its strategy except for mutation. If in period t two or more firms chose different strategies, at least one firm has a chance to change its strategy with positive probability without mutation, and all firms may choose the same strategy in period $t+1$. Thus, such a state cannot be included in a limit set, and in any state included in some limit set, all firms must choose the same strategy.[4] We need no mutation to move from any state, which is not included in a limit set, to a state in some limit set. Thus, a stochastically stable state must be in some limit set.

Denote the state in which all firms choose the output x by $\omega(x)$. The number of the states (including the state where $x=0$) is $\upsilon+1$. Denote the subset of Ω consisting of limit sets of **T** by Ω_l. Define an $\omega(x)$-tree as follows. An $\omega(x)$-tree is a function t: $\Omega_l \to \Omega_l$ such that $t(\omega(x)) = \omega(x)$ and such that for all $\omega \neq \omega(x)$, there exists m with $t^m(\omega) = \omega(x)$. We may think of an $\omega(x)$-tree as a set of arrows connecting elements of Ω_l in which every element of ω has a unique successor $t(\omega)$, and all paths eventually lead to $\omega(x)$. Define the cost of a move from ω to $t(\omega)$, $c(\omega, t(\omega))$, to be the minimum number of mutations needed to transit from ω to $t(\omega)$ under $\mathbf{T}_\varepsilon$, where $\mathbf{T}_\varepsilon$ is the transition matrix on Ω_l when the mutation probability is ε. Then, the cost of an ω-tree is the total cost of all moves in the tree:

$$\sum_{\omega \in \Omega_l} c(\omega, t(\omega)).$$

And we define $C(x)$ to be the minimum cost of all possible $\omega(x)$-trees. This is the minimum number of mutations needed to reach $\omega(x)$ from all the other limit sets. Based on the results of Freidlin and Wentzel (1984), in their Proposition 4, Kandori and Rob (1995) showed that the stochastically stable states comprise the states having minimum $C(x)$.

From the arguments in the previous section, we see that, since x^* is the GSS, only one mutation is sufficient to move to the state $\omega(x^*)$ from any other state. Therefore, $C(x^*) = \upsilon$. On the other hand, since x^* is also the finite population ESS, one mutation is not sufficient and we need at least two mutations to move from the state $\omega(x^*)$ to some other state. Thus, $C(x) \geqq \upsilon + 1$ for $x \neq x^*$. Hence,

[4] This result is similar to Proposition 1 in Vega-Redondo (1997).

x^* is the stochastically stable output. A transition to $\omega(x^*)$ from any other state occurs with only one mutation. On the other hand, a transition from $\omega(x^*)$ to some other state occurs with at least two mutations. Thus, the former transition is more probable than the latter. This is the reason why $\omega(x^*)$ is the stochastically stable state.

Next, consider the price-setting oligopoly. The analysis of the stochastically stable state in a price-setting oligopoly is parallel to the analysis in a quantity-setting oligopoly. We consider similar imitation dynamics. The firms must choose the prices of their goods from a finite grid $\Gamma' = \{0, \delta', 2\delta', \dots, v'\delta'\}$ where $\delta' > 0$ and $v' \in \mathbb{N}$ are arbitrary. It is required that the GSS (and the finite population ESS) price belongs to this grid.

In addition to the dynamic adjustment based on the comparison of the profits, there is a random mutation. In each period, each firm switches its strategy with probability ε. All strategies may be chosen with positive probability. Thus, the complete dynamic is an Ergodic Markov chain, and it has a unique stationary distribution. The stochastically stable states are the states that are assigned positive probability in the limit as $\varepsilon \to 0$.

Limit sets are similarly defined. We need no mutation to move from any state, which is not included in a limit set, to a state in some limit set. Denote the state in which all firms choose the price p by $\omega(p)$. The number of the states is $v' + 1$.

An $\omega(p)$-tree and $C(p)$ are similarly defined. From the arguments in the previous section, we see that, since p^* is the GSS, only one mutation is sufficient to move to the state $\omega(p^*)$ from any other state. Therefore, $C(p^*) = v'$. On the other hand, since p^* is also the finite population ESS, one mutation is not sufficient and we need at least two mutations to move from the state $\omega(p^*)$ to some other state. Therefore, $C(p) \geqq v' + 1$ for $p \neq p^*$ and, hence, p^* is the stochastically stable price.

In Theorem 1, we have shown that the GSS (and so the finite population ESS) in a quantity-setting oligopoly and that in a price-setting oligopoly are equivalent. Therefore, we obtain the following result.

Theorem 2. *The stochastically stable state in a quantity-setting oligopoly and that in a price-setting oligopoly coincide.*

We can invade any state other than the stochastically stable states by only one mutation with x^* or p^*, and can reach the stochastically stable states. Thus, *long run* in our model is not so long.

4. Interpretation

Let us consider the difference between stochastically stable strategies and Nash equilibrium strategies. In a quantity-setting oligopoly, when the goods are substitutes (or complements), each firm determines its output with a conjecture that if it increases its output, the prices of the other firms' goods will fall (or rise). Such responses reduce (or increase) the price of its good. On the other hand, in a price-setting oligopoly, each firm determines the price of its good with a conjecture that if it reduces the price of its good, the outputs (or demands) of the other firms will decrease (or increase). Such responses increase (or reduce) the demand for its good. Then, the firms in a price-setting oligopoly should be more (or less) aggressive than the firms in a quantity-setting oligopoly. These are why the Nash equilibrium in a quantity-setting oligopoly and that in a price-setting oligopoly are different.

With imitation dynamics, all that matters is that a quantity increase in a quantity-setting oligopoly and a price increase (or decrease) in a price-setting oligopoly; as long as it raises a firm's profit relatively to those of the other firms, it will be imitated. This suggests why both games lead to the same outcome.

We have defined a GSS as a strategy that, if adopted by a single mutant, can invade any configuration where the population monomorphically adopts some other strategy. On the other hand, a finite population ESS is defined as a strategy that is immune to the invasion of a single mutant. Suppose there exists a unique finite population ESS. If there exists a GSS that is different from the finite population ESS, then it is not immune to the invasion of a single mutant. Therefore, if there exists unique finite population ESS and GSS, they must be equal. But, there does not necessarily exist a GSS even though there exists a unique finite population ESS.

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