Maximin and minimax strategies in symmetric oligopoly

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Abstract

We examine maximin and minimax strategies for firms in duopoly with differentiated goods. We consider two patterns of game; the Cournot game in which strategic variables of the firms are their outputs, and the Bertrand game in which strategic variables of the firms are the prices of their goods. We call two firms Firm A and B, and will show that the maximin strategy and the minimax strategy in the Cournot game, and the maximin strategy and the minimax strategy in the Bertrand game are all equivalent for each firm. However, the maximin strategy (or the minimax strategy) for Firm A and that for Firm B are not necessarily equivalent, and they are not necessarily equivalent to their Nash equilibrium strategies in the Cournot game nor the Bertrand game.. But, in a special case, where the objective function of Firm B is the opposite of the objective function of Firm A, the maximin strategy for Firm A and that for Firm B are equivalent, and they constitute the Nash equilibrium both in the Cournot game and the Bertrand game. This special case corresponds to relative profit maximization by the firms.

keywords maximin strategy, minimax strategy, duopoly

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1 Introduction

We examine maximin and minimax strategies for firms in symmetric oligopoly with differentiated goods. We consider two patterns of game; the Cournot game in which strategic variables of the firms are their outputs, and the Bertrand game in which strategic variables of the firms are the prices of their goods. The maximin strategy for a firm is its strategy which maximizes its objective function that is minimized by a strategy of each rival firm. The minimax strategy for a firm is a strategy of each rival firm which minimizes its objective function that is maximized by its strategy. These strategies are defined for any pair of two firms. The objective functions of the firms may be or may not be their absolute profits. We will show that the maximin strategy and the minimax strategy in the Cournot game, and the maximin strategy and the minimax strategy in the Bertrand game for the firms are all equivalent. However, the maximin strategy (or the minimax strategy) for the firms are not necessarily equivalent to their Nash equilibrium strategies in the Cournot game nor the Bertrand game. But in a special case, where the objective function of one firm is the opposite of the sum of the objective functions of other firms, the maximin strategy (or the minimax strategy) for the firms constitute the Nash equilibrium both in the Cournot game and the Bertrand game, and in the special case the Nash equilibrium in the Cournot game and that in the Bertrand game are equivalent. This special case corresponds to relative profit maximization by the firms.

2 The model

There are n firms. Call each firm Firm i, $i \in \{1, 2, \dots, n\}$. The firms produce differentiated goods. The output and the price of the good of Firm i are denoted by x_i and p_i. The inverse demand functions are

$$p_i = f_i(x_1, x_2, \dots, x_n),\ i \in \{1, 2, \dots, n\}. \tag{1}$$

f_i's for all firms are the same functions. They are symmetric, continuous, differentiable and invertible. The inverses of them, that is, the direct demand functions are written as

$$x_i = g_i(p_1, p_2, \dots, p_n),\ i \in \{1, 2, \dots, n\}.$$

Differentiating (1) with respect to p_i given p_j, $j \in \{1, 2, \dots, n\}$, $j \neq i$, yields

$$\frac{\partial f_i}{\partial x_i}\frac{dx_i}{dp_i} + \sum_{j=1, j\neq i}^{n} \frac{\partial f_i}{\partial x_j}\frac{dx_j}{dp_i} = 1.$$

$$\frac{\partial f_j}{\partial x_i}\frac{dx_i}{dp_i} + \frac{\partial f_j}{\partial x_j}\frac{dx_j}{dp_i} + \sum_{k=1, k\neq i,j}^{n} \frac{\partial f_j}{\partial x_k}\frac{dx_k}{dp_i} = 0,\ j \in \{1, 2, \dots, n\},\ j \neq i.$$

By the symmetry of the model, since $\frac{\partial f_i}{\partial x_j}=\frac{\partial f_j}{\partial x_i}$ and $\frac{\partial f_j}{\partial x_j}=\frac{\partial f_i}{\partial x_i}$ at the equilibrium they are rewritten as

$$\frac{\partial f_i}{\partial x_i}\frac{dx_i}{dp_i}+(n-1)\frac{\partial f_j}{\partial x_i}\frac{dx_j}{dp_i}=1.$$

$$\frac{\partial f_j}{\partial x_i}\frac{dx_i}{dp_i}+\left[\frac{\partial f_i}{\partial x_i}+(n-2)\frac{\partial f_j}{\partial x_i}\right]\frac{dx_j}{dp_i}=0.$$

From them we get

$$\frac{dx_i}{dp_i}=\frac{dx_j}{dp_j}=\frac{\frac{\partial f_i}{\partial x_i}+(n-2)\frac{\partial f_j}{\partial x_i}}{\left(\frac{\partial f_i}{\partial x_i}-\frac{\partial f_j}{\partial x_i}\right)\left[\frac{\partial f_i}{\partial x_i}+(n-1)\frac{\partial f_j}{\partial x_i}\right]}, \tag{2}$$

$$\frac{dx_j}{dp_i}=\frac{dx_i}{dp_j}=-\frac{\frac{\partial f_j}{\partial x_i}}{\left(\frac{\partial f_i}{\partial x_i}-\frac{\partial f_j}{\partial x_i}\right)\left[\frac{\partial f_i}{\partial x_i}+(n-1)\frac{\partial f_j}{\partial x_i}\right]}, \tag{3}$$

because $\frac{dx_i}{dp_j}=\frac{dx_j}{dp_i}$ and $\frac{dx_i}{dp_i}=\frac{dx_j}{dp_j}$ at the equilibrium. We assume

$$\frac{\partial f_i}{\partial x_i}\neq 0,\ \frac{\partial f_j}{\partial x_i}\neq 0,\ \frac{\partial f_i}{\partial x_i}-\frac{\partial f_j}{\partial x_i}\neq 0,\ \frac{\partial f_i}{\partial x_i}+(n-1)\frac{\partial f_j}{\partial x_i}\neq 0,\ \frac{\partial f_i}{\partial x_i}+(n-2)\frac{\partial f_j}{\partial x_i}\neq 0. \tag{4}$$

The objective function of Firm $i,\ i\in\{1,2,\dots,n\}$ is

$$\pi_i(x_1,x_2,\dots,x_n).$$

It is continuous and differentiable. It may be or may not be the absolute profit of Firm i. We consider two patterns of game, the Cournot game and the Bertrand game. In the Cournot game strategic variables of the firms are their outputs, and in the Bertrand game their strategic variables are the prices of their goods. We do not consider simple maximization of their objective functions. Instead we investigate maximin strategies and minimax strategies for the firms.

3 Maximin and minimax strategies

3.1 Cournot game

3.1.1 Maximin strategy

First consider the condition for minimization of π_i with respect to $x_j,\ j\neq i$, given x_i and x_k's, $k\in\{1,2,\dots,n\},\ k\neq i,j$. It is

$$\frac{\partial \pi_i}{\partial x_j}=0,\ j\neq i. \tag{5}$$

Depending on the value of x_i we get the value of x_j which satisfies (5). Denote it by $x_j(x_i)$. Differentiating (5) with respect to x_i given x_k's $k \in \{1, 2, \dots, n\},\ k \neq i, j$,

$$\frac{\partial^2 \pi_i}{\partial x_i^2} + \frac{\partial^2 \pi_i}{\partial x_i \partial x_j} \frac{dx_j(x_i)}{dx_i} = 0.$$

From it

$$\frac{dx_j(x_i)}{dx_i} = -\frac{\frac{\partial^2 \pi_i}{\partial x_i^2}}{\frac{\partial^2 \pi_i}{\partial x_j \partial x_i}}.$$

We assume that it is not zero. The maximin strategy for Firm i is its strategy which maximizes π_i given $x_j(x_i)$ and x_k's $k \in \{1, 2, \dots, n\},\ k \neq i, j$. It is defined for any pair of i and $j \neq i$. The condition for maximization of π_i is

$$\frac{\partial \pi_i}{\partial x_i} + \frac{\partial \pi_i}{\partial x_j} \frac{dx_j(x_i)}{dx_i} = 0.$$

By (5) it is reduced to

$$\frac{\partial \pi_i}{\partial x_i} = 0.$$

Thus, the conditions for the maximin strategy Firm i are

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{6}$$

By the symmetry of the oligopoly the conditions in (6) are the same for all pairs of i and $j \neq i$.

3.1.2 Minimax strategy

Consider the condition for maximization of π_i with respect to x_i given $x_j,\ j \neq i$, and x_k's, $k \in \{1, 2, \dots, n\}, k \neq i, j$. It is

$$\frac{\partial \pi_i}{\partial x_i} = 0. \tag{7}$$

Depending on the value of x_j we get the value of x_i which satisfies (7). Denote it by $x_i(x_j)$. Differentiating (7) with respect to x_j given x_k's, $k \in \{1, 2, \dots, n\},\ k \neq i, j$.

$$\frac{\partial^2 \pi_i}{\partial x_i^2} \frac{dx_i}{dx_j} + \frac{\partial^2 \pi_i}{\partial x_j \partial x_i} = 0.$$

From it we obtain

$$\frac{dx_i(x_j)}{dx_j} = -\frac{\frac{\partial^2 \pi_i}{\partial x_j \partial x_i}}{\frac{\partial^2 \pi_i}{\partial x_i^2}}.$$

We assume that it is not zero. The minimax strategy for Firm i is a strategy of Firm $j,\ j \neq i$, which minimizes π_i given $x_i(x_j)$ and x_k's ,$k \in \{1, 2, \dots, n\},\ k \neq i, j$. It is defined for any pair of i and $j \neq i$. The condition for minimization of π_i is

$$\frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dx_j} + \frac{\partial \pi_i}{\partial x_j} = 0.$$

By (7) it is reduced to

$$\frac{\partial \pi_i}{\partial x_j} = 0.$$

Thus, the conditions for the minimax strategy for Firm i are

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}.$$

By the symmetry of the oligopoly these conditions are the same for all pairs of i and $j \neq i$. They are the same as conditions in (6).

3.2 Bertrand game

The objective function of Firm $i,\ i \in \{1, 2, \dots, n\}$, in the Bertrand game is written as follows.

$$\pi_i(x_1(p_1, p_2, \dots, p_n), x_2(p_1, p_2, \dots, p_n), \dots, x_n(p_1, p_2, \dots, p_n)).$$

We can write it as

$$\pi_i(p_1, p_2, \dots, p_n),$$

because π_i is a function of $p_1, p_2, \dots, p_n$. Exchanging x_i, x_j and x_k by p_i, p_j and p_k in the arguments in the previous subsection, we can show that the conditions for the maximin strategy and the minimax strategy for Firm i in the Bertrand game are as follows.

$$\frac{\partial \pi_i}{\partial p_i} = 0,\ \frac{\partial \pi_i}{\partial p_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{8}$$

By the symmetry of the oligopoly the conditions in (8) are the same for all pairs of i and $j \neq i$. We can rewrite them as follows.

$$\frac{\partial \pi_i}{\partial p_i} = \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dp_i} + (n-1)\frac{\partial \pi_i}{\partial x_j}\frac{dx_j}{dp_i} = 0,\ j \neq i,$$

$$\begin{aligned}\frac{\partial \pi_i}{\partial p_j} =& \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dp_j} + \frac{\partial \pi_i}{\partial x_j}\frac{dx_j}{dp_j} + (n-2)\frac{\partial \pi_i}{\partial x_k}\frac{dx_k}{dp_j} \\ =& \frac{\partial \pi_i}{\partial x_i}\frac{dx_i}{dp_j} + \frac{\partial \pi_i}{\partial x_j}\left[\frac{dx_i}{dp_i} + (n-2)\frac{dx_i}{dp_j}\right] = 0,\ j \neq i,\ k \neq i, j,\end{aligned}$$

because $\frac{dx_j}{dp_j} = \frac{dx_i}{dp_i}$, $\frac{\partial \pi_i}{\partial x_k} = \frac{\partial \pi_i}{\partial x_j}$ and $\frac{dx_k}{dp_j} = \frac{dx_i}{dp_j}$. By (2) and (3) and the assumptions in (4), they are further rewritten as

$$\frac{\partial \pi_i}{\partial x_i}\left[\frac{\partial f_i}{\partial x_i} + (n-2)\frac{\partial f_j}{\partial x_i}\right] - (n-1)\frac{\partial \pi_i}{\partial x_j}\frac{\partial f_j}{\partial x_i} = 0,$$

$$\frac{\partial \pi_i}{\partial x_i}\frac{\partial f_j}{\partial x_i} - \frac{\partial \pi_i}{\partial x_j}\frac{\partial f_i}{\partial x_i} = 0.$$

Again by the assumptions in (4), we obtain

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_j} = 0,\ j \neq i.$$

They are the same as conditions in (6). We have proved the following proposition.

Proposition 1. *The maximin strategy and the minimax strategy in the Cournot game, and the maximin strategy and the minimax strategy in the Bertrand game for the firms are all equivalent.*

4 Special case

The results in the previous section do not imply that the maximin strategies (or the minimax strategies) for the firms are equivalent to their Nash equilibrium strategies in the Cournot game nor the Bertrand game. But in a special case the maximin strategies (or the minimax strategies) for the firms constitute the Nash equilibrium both in the Cournot game and the Bertrand game.

The conditions for the maximin strategy and the minimax strategy for the firms are

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{6}$$

The conditions for Nash equilibrium in the Cournot game are

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_j}{\partial x_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{9}$$

(6) and (9) are not necessarily equivalent. The conditions for Nash equilibrium in the Bertrand game are

$$\frac{\partial \pi_i}{\partial p_i} = 0,\ \frac{\partial \pi_j}{\partial p_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{10}$$

(8) and (10) are not necessarily equivalent.

However, in a special case those conditions are all equivalent. We assume

$$\pi_i = -\sum_{j=1,j\neq i}^{n} \pi_j,\ \text{or}\ \pi_i + \sum_{j=1,j\neq i}^{n} \pi_j = 0. \tag{11}$$

By the symmetry of the oligopoly

$$\pi_i = -(n-1)\pi_j.$$

Then, (9) is rewritten as

$$\frac{\partial \pi_i}{\partial x_i} = 0,\ \frac{\partial \pi_i}{\partial x_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{12}$$

(12) and (6) are equivalent. Therefore, the maximin strategies and the minimax strategies for the firms in the Cournot game constitute the Nash equilibrium of the Cournot game. $\frac{\partial \pi_j}{\partial x_j} = 0$ in (9) means maximization of π_j with respect to x_j. On the other hand, $\frac{\partial \pi_i}{\partial x_j} = 0$ in (12) means minimization of π_i with respect to x_j.

Similarly, (10) is rewritten as

$$\frac{\partial \pi_i}{\partial p_i} = 0,\ \frac{\partial \pi_i}{\partial p_j} = 0,\ j \neq i,\ i \in \{1, 2, \dots, n\}. \tag{13}$$

(13) and (8) are equivalent. Therefore, the maximin strategies and the minimax strategies for the firms in the Bertrand game constitute the Nash equilibrium of the Bertrand game. Since the maximin strategies and the minimax strategies in the Cournot game and those in the Bertrand game are equivalent, the Nash equilibrium of the Cournot game and that of the Bertrand game are equivalent.

Summarizing the results, we get the following proposition.

Proposition 2. *In the special case in which (11) is satisfied: The maximin strategies and the minimax strategies for the firms constitute the Nash equilibrium both in the Cournot game and the Bertrand game.*

This special case corresponds to relative profit maximization[1]. Let $\bar{\pi}_i$ be the absolute profit of Firm i, $i \in \{1, 2, \dots, n\}$, and denote its relative profit by π_i. Then,

$$\pi_i = \bar{\pi}_i - \frac{1}{n-1}\sum_{j=1,j\neq i}^{n} \bar{\pi}_j,\ i \in \{1, 2, \dots, n\}.$$

We have

$$\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \bar{\pi}_i - \sum_{i=1}^{n} \bar{\pi}_i = 0.$$

By the symmetry of the oligopoly

$$\pi_i = -(n-1)\pi_j.$$

[1] About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997).

5 Concluding Remark

In the future research we want to extend arguments of this paper to a case where we do not postulate differentiability of objective functions[2]

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[2] One attempt along this line is Satoh and Tanaka (2016).