Maximin and minimax strategies in asymmetric duopoly: Cournot and Bertrand

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Abstract

We examine maximin and minimax strategies for firms in asymmetric duopoly with differentiated goods. We consider two patterns of game; the Cournot game in which strategic variables of the firms are their outputs, and the Bertrand game in which strategic variables of the firms are the prices of their goods. We call two firms Firm A and B, and will show that the maximin strategy and the minimax strategy in the Cournot game, and the maximin strategy and the minimax strategy in the Bertrand game are all equivalent for each firm. However, the maximin strategy (or the minimax strategy) for Firm A and that for Firm B are not necessarily equivalent, and they are not necessarily equivalent to their Nash equilibrium strategies in the Cournot game nor the Bertrand game.. But, in a special case, where the objective function of Firm B is the opposite of the objective function of Firm A, the maximin strategy for Firm A and that for Firm B are equivalent, and they constitute the Nash equilibrium both in the Cournot game and the Bertrand game. This special case corresponds to relative profit maximization by the firms.

keywords maximin strategy, minimax strategy, duopoly

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1 Introduction

We examine maximin and minimax strategies for firms in duopoly with differentiated goods. We consider two patterns of game; the Cournot game in which strategic variables of the firms are their outputs, and the Bertrand game in which strategic variables of the firms are the prices of their goods. The maximin strategy for a firm is its strategy which maximizes its objective function that is minimized by a strategy of its rival firm. The minimax strategy for a firm is a strategy of its rival firm which minimizes its objective function that is maximized by its strategy. The objective functions of the firms may be or may not be their absolute profits. We call two firms Firm A and B, and will show the following results.

(1) The maximin strategy and the minimax strategy in the Cournot game, and the maximin strategy and the minimax strategy in the Bertrand game for Firm A are all equivalent.

(2) The maximin strategy and the minimax strategy in the Cournot game, and the maximin strategy and the minimax strategy in the Bertrand game for Firm B are all equivalent.

However, the maximin strategy (or the minimax strategy) for Firm A and that for Firm B are not necessarily equivalent (if the duopoly is not symmetric), and they are not necessarily equivalent to their Nash equilibrium strategies in the Cournot game nor the Bertrand game[1]. But in a special case, where the objective function of Firm B is the opposite of the objective function of Firm A, the maximin strategy (or the minimax strategy) for Firm A and that for Firm B are equivalent, and they constitute the Nash equilibrium both in the Cournot game and the Bertrand game. Thus, in the special case the Nash equilibrium in the Cournot game and that in the Bertrand game are equivalent. This special case corresponds to relative profit maximization by the firms.

In the appendix we consider a mixed game in which one of the firms chooses the output and the other firm chooses the price as their strategic variables, and show that the maximin and minimax strategies for each firm in the mixed game are equivalent to those in the Cournot game.

2 The model

There are two firms, Firm A and B. They produce differentiated goods. The outputs and the prices of the goods are denoted by x_A and p_A for Firm A, and x_B and p_B for Firm B. The inverse demand functions are

$$p_A = f_A(x_A, x_B),\ p_B = f_B(x_A, x_B). \tag{1}$$

They are continuous, differentiable and invertible. The inverses of them, that is, the direct demand functions are written as

$$x_A = g_A(p_A, p_B),\ x_B = g_B(p_A, p_B).$$

[1] If the duopoly is symmetric, the maximin strategy (or the minimax strategy) for Firm A and that for Firm B are equivalent. But even if the duopoly is symmetric, they are not necessarily equivalent to their Nash equilibrium strategies.

Differentiating (1) with respect to p_A given p_B yields

$$\frac{\partial f_A}{\partial x_A}\frac{dx_A}{dp_A}+\frac{\partial f_A}{\partial x_B}\frac{dx_B}{dp_A}=1,\ \frac{\partial f_B}{\partial x_A}\frac{dx_A}{dp_A}+\frac{\partial f_B}{\partial x_B}\frac{dx_B}{dp_A}=0.$$

From them we get

$$\frac{dx_A}{dp_A}=\frac{\frac{\partial f_B}{\partial x_B}}{\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_A}{\partial x_B}\frac{\partial f_B}{\partial x_A}},\ \frac{dx_B}{dp_A}=-\frac{\frac{\partial f_B}{\partial x_A}}{\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_A}{\partial x_B}\frac{\partial f_B}{\partial x_A}}. \tag{2}$$

Symmetrically,

$$\frac{dx_B}{dp_B}=\frac{\frac{\partial f_A}{\partial x_A}}{\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_A}{\partial x_B}\frac{\partial f_B}{\partial x_A}},\ \frac{dx_A}{dp_B}=-\frac{\frac{\partial f_A}{\partial x_B}}{\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_A}{\partial x_B}\frac{\partial f_B}{\partial x_A}}. \tag{3}$$

We assume

$$\frac{\partial f_A}{\partial x_A}\neq 0,\ \frac{\partial f_B}{\partial x_B}\neq 0,\ \frac{\partial f_A}{\partial x_B}\neq 0,\ \frac{\partial f_B}{\partial x_A}\neq 0,\ \left|\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_A}{\partial x_B}\frac{\partial f_B}{\partial x_A}\right|\neq 0. \tag{4}$$

The objective functions of Firm A and B are

$$\pi_A(x_A,x_B),\pi_B(x_A,x_B).$$

They are continuous and differentiable. They may be or may not be the absolute profits of the firms. We consider two patterns of game, the Cournot game and the Bertrand game. In the Cournot game strategic variables of the firms are their outputs, and in the Bertrand game their strategic variables are the prices of their goods. We do not consider simple maximization of their objective functions. Instead we investigate maximin strategies and minimax strategies for the firms.

3 Maximin and minimax strategies

3.1 Cournot game

3.1.1 Maximin strategy

First consider the condition for minimization of π_A with respect to x_B. It is

$$\frac{\partial \pi_A}{\partial x_B}=0. \tag{5}$$

Depending on the value of x_A we get the value of x_B which satisfies (5). Denote it by $x_B(x_A)$. From (5)

$$\frac{dx_B(x_A)}{dx_A}=-\frac{\frac{\partial^2\pi_A}{\partial x_A\partial x_B}}{\frac{\partial^2\pi_A}{\partial x_B^2}}.$$

We assume that it is not zero. The maximin strategy for Firm A is its strategy which maximizes $\pi_A(x_A, x_B(x_A))$. The condition for maximization of $\pi_A(x_A, x_B(x_A))$ with respect to x_A is

$$\frac{\partial \pi_A}{\partial x_A} + \frac{\partial \pi_A}{\partial x_B}\frac{dx_B(x_A)}{dx_A} = 0.$$

By (5) it is reduced to

$$\frac{\partial \pi_A}{\partial x_A} = 0.$$

Thus, the conditions for the maximin strategy for Firm A are

$$\frac{\partial \pi_A}{\partial x_A} = 0,\ \frac{\partial \pi_A}{\partial x_B} = 0. \tag{6}$$

3.1.2 Minimax strategy

Consider the condition for maximization of π_A with respect to x_A. It is

$$\frac{\partial \pi_A}{\partial x_A} = 0. \tag{7}$$

Depending on the value of x_B we get the value of x_A which satisfies (7). Denote it by $x_A(x_B)$. From (7) we obtain

$$\frac{dx_A(x_B)}{dx_B} = -\frac{\frac{\partial^2 \pi_A}{\partial x_B \partial x_A}}{\frac{\partial^2 \pi_A}{\partial x_A^2}}.$$

We assume that it is not zero. The minimax strategy for Firm A is a strategy of Firm B which minimizes $\pi_A(x_A(x_B), x_B)$. The condition for minimization of $\pi_A(x_A(x_B), x_B)$ with respect to x_B is

$$\frac{\partial \pi_A}{\partial x_A}\frac{dx_A(x_B)}{dx_B} + \frac{\partial \pi_A}{\partial x_B} = 0.$$

By (7) it is reduced to

$$\frac{\partial \pi_A}{\partial x_B} = 0.$$

Thus, the conditions for the maximin strategy for Firm A are

$$\frac{\partial \pi_A}{\partial x_A} = 0,\ \frac{\partial \pi_A}{\partial x_B} = 0.$$

They are the same as conditions in (6). Similarly, we can show that the conditions for the maximin strategy and the minimax strategy for Firm B are

$$\frac{\partial \pi_B}{\partial x_B} = 0,\ \frac{\partial \pi_B}{\partial x_A} = 0. \tag{8}$$

3.2 Bertrand game

The objective functions of Firm A and B in the Bertrand game are written as follows.

$$\pi_A(x_A(p_A, p_B), x_B(p_A, p_B)),\ \pi_B(x_A(p_A, p_B), x_B(p_A, p_B)).$$

We can write them as

$$\pi_A(p_A, p_B),\ \pi_B(p_A, p_B),$$

because $\pi_A(x_A(p_A, p_B), x_B(p_A, p_B))$ and $\pi_B(x_A(p_A, p_B), x_B(p_A, p_B))$ are functions of p_A and p_B. Exchanging x_A and x_B by p_A and p_B in the arguments in the previous subsection, we can show that the conditions for the maximin strategy and the minimax strategy for Firm A in the Bertrand game are as follows.

$$\frac{\partial \pi_A}{\partial p_A} = 0,\ \frac{\partial \pi_A}{\partial p_B} = 0. \tag{9}$$

We can rewrite them as follows.

$$\frac{\partial \pi_A}{\partial x_A}\frac{dx_A}{dp_A} + \frac{\partial \pi_A}{\partial x_B}\frac{dx_B}{dp_A} = 0,\ \frac{\partial \pi_A}{\partial x_A}\frac{dx_A}{dp_B} + \frac{\partial \pi_A}{\partial x_B}\frac{dx_B}{dp_B} = 0.$$

By (2) and (3) and the assumptions in (4), they are further rewritten as

$$\frac{\partial \pi_A}{\partial x_A}\frac{\partial f_B}{\partial x_B} - \frac{\partial \pi_A}{\partial x_B}\frac{\partial f_B}{\partial x_A} = 0,\ \frac{\partial \pi_A}{\partial x_A}\frac{\partial f_A}{\partial x_B} - \frac{\partial \pi_A}{\partial x_B}\frac{\partial f_A}{\partial x_A} = 0.$$

Again by the assumptions in (4), we obtain

$$\frac{\partial \pi_A}{\partial x_A} = 0,\ \frac{\partial \pi_A}{\partial x_B} = 0.$$

They are the same as conditions in (6).

The conditions for the maximin strategy and the minimax strategy for Firm B in the Bertrand game are

$$\frac{\partial \pi_B}{\partial p_B} = 0,\ \frac{\partial \pi_B}{\partial p_A} = 0.$$

They are rewritten as

$$\frac{\partial \pi_B}{\partial x_B}\frac{dx_B}{dp_B} + \frac{\partial \pi_B}{\partial x_A}\frac{dx_A}{dp_B} = 0,\ \frac{\partial \pi_B}{\partial x_B}\frac{dx_B}{dp_A} + \frac{\partial \pi_B}{\partial x_A}\frac{dx_A}{dp_A} = 0.$$

By (2) and (3) and the assumptions in (4), they are further rewritten as

$$\frac{\partial \pi_B}{\partial x_B}\frac{\partial f_A}{\partial x_A} - \frac{\partial \pi_B}{\partial x_A}\frac{\partial f_A}{\partial x_B} = 0,\ \frac{\partial \pi_B}{\partial x_B}\frac{\partial f_B}{\partial x_A} - \frac{\partial \pi_B}{\partial x_A}\frac{\partial f_B}{\partial x_B} = 0.$$

Again by the assumptions in (4), we obtain

$$\frac{\partial \pi_B}{\partial x_A} = 0,\ \frac{\partial \pi_B}{\partial x_B} = 0.$$

They are the same as conditions in (8). We have proved the following proposition.

Proposition 1. *(1) The maximin strategy and the minimax strategy in the Cournot game, and the maximin strategy and the minimax strategy in the Bertrand game for Firm A are all equivalent.*

(2) The maximin strategy and the minimax strategy in the Cournot game, and the maximin strategy and the minimax strategy in the Bertrand game for Firm B are all equivalent.

4 Special case

The results in the previous section do not imply that the maximin strategy (or the minimax strategy) for Firm A and that for Firm B are equivalent (if the duopoly is not symmetric), and they are equivalent to their Nash equilibrium strategies in the Cournot game nor the Bertrand game. But in a special case the maximin strategy (or the minimax strategy) for Firm A and that for Firm B are equivalent, and they constitute the Nash equilibrium both in the Cournot game and the Bertrand game.

The conditions for the maximin strategy and the minimax strategy for Firm A are

$$\frac{\partial \pi_A}{\partial x_A} = 0,\ \frac{\partial \pi_A}{\partial x_B} = 0. \tag{6}$$

Those for Firm B are

$$\frac{\partial \pi_B}{\partial x_B} = 0,\ \frac{\partial \pi_B}{\partial x_A} = 0. \tag{8}$$

(6) and (8) are not necessarily equivalent. The conditions for Nash equilibrium in the Cournot game are

$$\frac{\partial \pi_A}{\partial x_A} = 0,\ \frac{\partial \pi_B}{\partial x_B} = 0. \tag{10}$$

(6) and (10) are not necessarily equivalent. The conditions for Nash equilibrium in the Bertrand game are

$$\frac{\partial \pi_A}{\partial p_A} = 0,\ \frac{\partial \pi_B}{\partial p_B} = 0. \tag{11}$$

(9) and (11) are not necessarily equivalent.

However, in a special case those conditions are all equivalent. We assume

$$\pi_B = -\pi_A \text{ or } \pi_A + \pi_B = 0. \tag{12}$$

Then, (8) is rewritten as

$$\frac{\partial \pi_A}{\partial x_B} = 0,\ \frac{\partial \pi_A}{\partial x_A} = 0. \tag{13}$$

They are equivalent to (6). Therefore, the maximin strategy and the minimax strategy for Firm A and those for Firm B are equivalent. $\frac{\partial \pi_B}{\partial x_A} = 0$ and $\frac{\partial \pi_B}{\partial x_B} = 0$ in (8) mean, respectively, minimization of π_B with respect to x_A and maximization of π_B with respect to x_B. On the other hand, $\frac{\partial \pi_A}{\partial x_A} = 0$ and $\frac{\partial \pi_A}{\partial x_B} = 0$ in (6) and (13) mean, respectively, maximization of π_A with respect to x_A and minimization of π_A with respect to x_B.

(10) is rewritten as

$$\frac{\partial \pi_A}{\partial x_A} = 0,\ \frac{\partial \pi_A}{\partial x_B} = 0. \tag{14}$$

(14) and (6) are equivalent. Therefore, the maximin strategy (Firm A's strategy) and the minimax strategy (Firm B's strategy) for Firm A constitute the Nash equilibrium of the Cournot game. $\frac{\partial \pi_B}{\partial x_B} = 0$ in (10) means maximization of π_B with respect to x_B. On the other hand, $\frac{\partial \pi_A}{\partial x_B} = 0$ in (14) means minimization of π_A with respect to x_B.

(11) is rewritten as

$$\frac{\partial \pi_A}{\partial p_A} = 0,\ \frac{\partial \pi_A}{\partial p_B} = 0. \tag{15}$$

(15) and (9) are equivalent. Therefore, the maximin strategy (Firm A's strategy) and the minimax strategy (Firm B's strategy) for Firm A in the Bertrand game constitute the Nash equilibrium of the Bertrand game. Since the maximin strategy and the minimax strategy for Firm A in the Cournot game and those in the Bertrand game are equivalent, the Nash equilibrium of the Cournot game and that of the Bertrand game are equivalent.

Summarizing the results, we get the following proposition.

Proposition 2. *In the special case in which (12) is satisfied:*

(1) *The maximin strategy and the minimax strategy in the Cournot game and the Bertrand game for Firm A and the maximin strategy and the minimax strategy in the Cournot game and the Bertrand game for Firm B are equivalent.*

(2) *These maximin and minimax strategies constitute the Nash equilibrium both in the Cournot game and the Bertrand game.*

This special case corresponds to relative profit maximization[2]. Let $\bar{\pi}_A$ and $\bar{\pi}_B$ be the absolute profits of Firm A and B, and denote their relative profits by π_A and π_B. Then,

$$\pi_A = \bar{\pi}_A - \bar{\pi}_B,\ \pi_B = \bar{\pi}_B - \bar{\pi}_A.$$

From them we can see

$$\pi_B = -\pi_A.$$

5 Concluding Remark

We have analyzed maximin and minimax strategies in Cournot and Bertrand games in duopoly. We assumed differentiability of objective functions of firms. In the future research we want to extend arguments of this paper to a case where we do not postulate differentiability of objective functions[3] and to a case of symmetric oligopoly with more than two firms.

[2] About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997).

[3] One attempt along this line is Satoh and Tanaka (2016).

Appendix: Mixed game

We consider a case where Firm A's strategic variable is p_A, and that of Firm B is x_B.

Differentiating (1) with respect to p_A given x_B yields

$$\frac{\partial f_A}{\partial x_A}\frac{dx_A}{dp_A}=1,\ \frac{\partial f_B}{\partial x_A}\frac{dx_A}{dp_A}=\frac{dp_B}{dp_A}.$$

Differentiating (1) with respect to x_B given p_A yields

$$\frac{\partial f_A}{\partial x_A}\frac{dx_A}{dx_B}+\frac{\partial f_A}{\partial x_B}=0,\ \frac{\partial f_B}{\partial x_A}\frac{dx_A}{dx_B}+\frac{\partial f_B}{\partial x_B}=\frac{dp_B}{dx_B}.$$

From them we obtain

$$\frac{dx_A}{dp_A}=\frac{1}{\frac{\partial f_A}{\partial x_A}},\ \frac{dp_B}{dp_A}=\frac{\frac{\partial f_B}{\partial x_A}}{\frac{\partial f_A}{\partial x_A}},\ \frac{dx_A}{dx_B}=-\frac{\frac{\partial f_A}{\partial x_B}}{\frac{\partial f_A}{\partial x_A}},\ \frac{dp_B}{dx_B}=\frac{\frac{\partial f_A}{\partial x_A}\frac{\partial f_B}{\partial x_B}-\frac{\partial f_B}{\partial x_A}\frac{\partial f_A}{\partial x_B}}{\frac{\partial f_A}{\partial x_A}}.$$

We assume $\frac{dx_A}{dp_A}\neq 0$ and $\frac{\partial f_A}{\partial x_B}\neq 0$, and so $\frac{dx_A}{dx_B}\neq 0$.

We write the objective functions of Firm A and B as follows.

$$\varphi_A(p_A,x_B)=\pi_A(x_A(p_A,p_B),x_B),\ \varphi_B(p_A,x_B)=\pi_B(x_A(p_A,p_B),x_B).$$

Then,

$$\begin{cases}\frac{\partial\varphi_A}{\partial p_A}=\frac{\partial\pi_A}{\partial x_A}\frac{dx_A}{dp_A},\ \frac{\partial\varphi_A}{\partial x_B}=\frac{\partial\pi_A}{\partial x_A}\frac{dx_A}{dx_B}+\frac{\partial\pi_A}{\partial x_B},\\ \frac{\partial\varphi_B}{\partial p_A}=\frac{\partial\pi_B}{\partial x_A}\frac{dx_A}{dp_A},\ \frac{\partial\varphi_B}{\partial x_B}=\frac{\partial\pi_B}{\partial x_A}\frac{dx_A}{dx_B}+\frac{\partial\pi_B}{\partial x_B}.\end{cases}\tag{16}$$

By similar ways to arguments in Section 3, we can show that for Firm A the conditions for the maximin strategy and the conditions for the minimax strategy are equivalent, and they are

$$\frac{\partial\varphi_A}{\partial p_A}=0,\ \frac{\partial\varphi_A}{\partial x_B}=0.\tag{17}$$

For Firm B the conditions for the maximin strategy and the minimax strategy are

$$\frac{\partial\varphi_A}{\partial p_A}=0,\ \frac{\partial\varphi_A}{\partial x_B}=0.\tag{18}$$

By (16), (17) is rewritten as

$$\frac{\partial\pi_A}{\partial x_A}\frac{dx_A}{dp_A}=0,\ \frac{\partial\pi_A}{\partial x_A}\frac{dx_A}{dx_B}+\frac{\partial\pi_A}{\partial x_B}=0.$$

Similarly, (18) is rewritten as follows.

$$\frac{\partial\pi_B}{\partial x_A}\frac{dx_A}{dp_A}=0,\ \frac{\partial\pi_B}{\partial x_A}\frac{dx_A}{dx_B}+\frac{\partial\pi_B}{\partial x_B}=0.$$

By the assumption $\frac{dx_A}{dp_A} \neq 0$ and $\frac{dx_A}{dx_B} \neq 0$, we obtain

$$\frac{\partial \pi_A}{\partial x_A} = 0,\ \frac{\partial \pi_A}{\partial x_B} = 0,$$

and

$$\frac{\partial \pi_B}{\partial x_A} = 0,\ \frac{\partial \pi_B}{\partial x_B} = 0.$$

They are the same as the conditions for the maximin and minimax strategies for Firm A and B in the Cournot game. Therefore, the maximin strategy and the minimax strategy for each firm in the mixed game are equivalent to those in the Cournot game.

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