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On the equivalence of the Arrow impossibility theorem and the Brouwer fixed point theorem when individual preferences are weak orders

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ABSTRACT

We will show that in the case where there are two individuals and three alternatives (or under the assumption of the free-triple property), and individual preferences are weak orders (which may include indifference relations), the Arrow impossibility theorem [Arrow, K.J., 1963. Social Choice and Individual Values, second ed. Yale University Press] that there exists no binary social choice rule which satisfies the conditions of transitivity, Pareto principle, independence of irrelevant alternatives, and non-existence of dictator is equivalent to the Brouwer fixed point theorem on a 2-dimensional ball (circle). Our study is an application of ideas by Chichilnisky [Chichilnisky, G., 1979. On fixed points and social choice paradoxes. Economics Letters 3, 347–351] to a discrete social choice problem, and also it is in line with the work by Baryshnikov [Baryshnikov, Y., 1993. Unifying impossibility theorems: a topological approach. Advances in Applied Mathematics 14, 404–415].

1. Introduction

Topological approaches to social choice problems have been initiated by Chichilnisky (1980). In her model a space of alternatives is a subset of a Euclidean space, and individual preferences over this set are represented by normalized gradient fields. Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. This approach has been further developed by Chichilnisky (1979, 1982), Koshevoy (1997), Lauwers (2004), Weinberger (2004), and so on. In particular, by Chichilnisky (1979) the equivalence of her impossibility result and the Brouwer fixed point theorem in the case where there are two individuals and the choice space is a 2-dimensional Euclidian space has been shown. On the other hand, Baryshnikov (1993, 1997) have presented a topological approach to Arrow's general possibility theorem, which is usually called the *Arrow impossibility theorem* (Arrow, 1963), in a discrete framework of social choice.[1]

We will examine the relation between the Arrow impossibility theorem that there exists no binary social choice rule that satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator,[2] and the Brouwer fixed point theorem on a 2-dimensional ball in the case where there are two individuals and three alternatives (or under the assumption of the free-triple property), and individual preferences are weak orders, which may include indifference

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[1] About surveys and basic results of topological social choice theories, see Mehta (1997) and Lauwers (2000). The book by Heal (1997) contains many recent papers in this field. Baryshnikov (1997) and Mehta (1997) are contained in this book.

[2] The dictator is an individual whose strict preference about each pair of two alternatives always coincides with the social preference.

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relations.[3] Our study is an application of ideas by Chichilnisky (1979) to a discrete social choice problem, and also it is in line with the work by Baryshnikov (1993). Our main tool is the homomorphism of homology groups of simplicial complexes induced by simplicial mappings. Of course, the Brouwer fixed point theorem is a theorem about continuous functions. We will consider a method to obtain a continuous function from a discrete social choice rule. Mainly we will show the following results.

(1) The Brouwer fixed point theorem is equivalent to the proposition that the restriction to an $(n-1)$-dimensional sphere S^{n-1} of a continuous function from an n-dimensional ball D^n to S^{n-1} is homotopic to a constant mapping (Theorem 1).
(2) The restriction of a continuous function obtained from a binary social choice rule, which satisfies transitivity, Pareto principle, independence of irrelevant alternatives (IIA) and has no dictator, to a subset of the set of profiles of individual preferences, which is homeomorphic to a 2-dimensional ball (circle) and the subset is homeomorphic to a 1-dimensional sphere (circumference), is not homotopic to any constant mapping (Lemma 1 and 2).
(3) Then, the non-existence of binary social choice rule which satisfies transitivity, Pareto principle, IIA and has no dictator is equivalent to the Brouwer fixed point theorem on a 2-dimensional ball (Theorem 2).

The second result is the main contribution of this paper. Intuitive explanations for it are as follows.

> The subset of the set of profiles of individual preferences is denoted by $\Delta \times A \times B$, which is homeomorphic to a circumference. Δ is the set of profiles such that two individuals have the same preference. A (or B) is the set of profiles such that the preference of one individual (individual A or B) is fixed at some preference, which is common to A and B. By Pareto principle the strict preferences represented by the vertices of Δ correspond to the same social preference by any binary social choice rule. And, considering barycentric subdivisions of simplicial complexes of Δ which include vertices that represent individual preferences including indifference relations, we can show that a continuous function obtained from a binary social choice rule from Δ to S^1, which is the simplicial complex that represents the social preference and is homeomorphic to a circumference, is surjective (onto mapping).
> On the other hand, non-existence of dictator and IIA with Pareto principle imply that the preferences represented by the vertices of A (or B) correspond to the fixed preference of individual A (or B). Thus, the continuous function from $A \times B$ to S^1 is a constant mapping.
> Therefore, the continuous function from $\Delta \times A \times B$ to S^1 is surjective, and its degree is not zero.

In the next section we present the model and the expression of a social choice rule using simplicial complexes. In Section 3 we will show a result about the Brouwer fixed point theorem and homotopy of continuous functions. In Section 4 we will prove the main results.

2. The model and simplicial complexes

We consider a case where there are two individuals, A and B, and three alternatives of some social problem, x_1, x_2 and x_3. Or we assume the free-triple property for n ($n \geqq 3$) alternatives. The free-triple property means that for each set of three alternatives, the preferences of individuals are never restricted. Individual preferences for the alternatives are weak orders, that is, individuals prefer one alternative to another, or are indifferent between them. These preferences must be complete and transitive. A social choice rule that we will consider is a rule which determines a social preference about the alternatives corresponding to a combination of the preferences of two individuals. The binary social choice rule which satisfies transitivity is called a *social welfare function*. We require that social welfare functions satisfy Pareto principle and independence of irrelevant alternatives (IIA). The meanings of the latter two conditions are as follows.

Pareto principle: If all individuals prefer an alternative x_i to another alternative x_j, then the society should prefer x_i to x_j.
Independence of irrelevant alternatives (IIA): The social preference about any pair of two alternatives x_i and x_j is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x_i and x_j.

The Arrow impossibility theorem states that there exists a dictator for any social welfare function which satisfies Pareto principle and IIA, or in other words there exists no social welfare function which satisfies these conditions and has no dictator. The dictator is an individual whose strict preference always coincides with the social preference.

From the set of individual preferences about three alternatives x_1, x_2 and x_3 we draw a diagram by the following procedures.

[3] In Tanaka (2006) we have shown that when individual preferences over alternatives are linear (strict), the Arrow impossibility theorem and the Brouwer fixed point theorem are equivalent. This paper will extend this result to the case where individual preferences are weak orders. The proof in the case of linear preferences is considerably simpler than that in the case of weak orders. In the former case we do not need consider any subdivision of simplicial complexes.

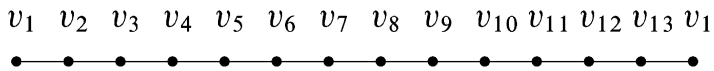

Fig. 1. *R*.

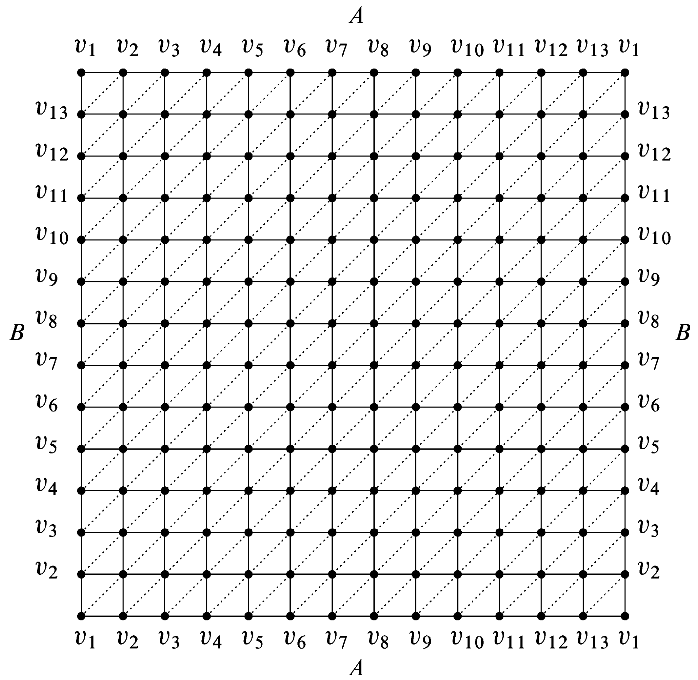

Fig. 2. $R \times R$.

(1) When an individual prefers x_1 to x_2 to x_3, such a preference is denoted by (123), and corresponding to this preference we define a vertex v_1. Similarly, when an individual prefers x_2 to x_3 and is indifferent between x_1 and x_2, such a preference is denoted by $(\overline{12}3)$, and we define a vertex v_2. And, when an individual prefers x_2 to x_1 to x_3, such a preference is denoted by (213), and we define a vertex v_3. By similar procedures the following vertices are defined.

$$v_1 = (123),\ v_2 = (\overline{12}3),\ v_3 = (213),\ v_4 = (2\overline{13}),\ v_5 = (231),\ v_6 = (\overline{23}1),\ v_7 = (321),$$

$$v_8 = (3\overline{12}),\ v_9 = (312),\ v_{10} = (\overline{13}2),\ v_{11} = (132),\ v_{12} = (1\overline{23}),\ v_{13} = (\overline{123})$$

For example, $v_7 = (321)$ denotes a preference of an individual such that he prefers x_3 to x_2 to x_1, and $v_{13} = (\overline{123})$ denotes a preference of an individual such that he is indifferent about all of three alternatives.

(2) These 13 vertices are plotted on a line segment in this order, locate v_1 at both end points, and connect the vertices.

Denote this diagram by R, and call $v_1, v_2, \ldots, v_{13}$ the vertices of R. It is depicted in Fig. 1.

The set of individual preferences is represented by R, and the set of combinations of the preferences of two individuals is represented by the product space $R \times R$. These combinations are called *profiles*. $R \times R$ is depicted as a square in Fig. 2. The preference of individual B is represented from bottom up, not from left to right. Individual preferences are denoted by $p_A = v_1$, $p_B = v_2$ and so on, and profiles are denoted by $\mathbf{p} = (p_A, p_B) = (v_1, v_3)$, and so on.

The social preference is represented by a circumference depicted in Fig. 3. We call the circumference S^1. The vertices in S^1 are denoted by $w_1, w_2, \ldots, w_{13}$. These vertices of S^1 represent the following social preferences.

$$w_1 = (123),\ w_2 = (\overline{12}3),\ w_3 = (213),\ w_4 = (2\overline{13}),\ w_5 = (231),\ w_6 = (\overline{23}1),\ w_7 = (321),\ w_8 = (3\overline{12}),\ w_9 = (312),$$

$$w_{10} = (\overline{13}2),\ w_{11} = (132),\ w_{12} = (1\overline{23}),\ w_{13} = (\overline{123})$$

For example, $w_3 = (213)$ denotes a preference such that the society prefers x_2 to x_1 to x_3, and $w_{13} = (\overline{123})$ denotes a preference such that the society is indifferent about all of three alternatives.

The 1-dimensional homology group of S^1 is isomorphic to the group of integers $\mathbb{Z}$, that is,[4] we have $H_1(S^1) \cong \mathbb{Z}$.

A social welfare function F is defined as a function from the vertices of $R \times R$ to the vertices of S^1. Let us consider a method to obtain a continuous function from a social welfare function defined on the vertices of $R \times R$. For example, for points

[4] About homology groups we referred to Tamura (1970) and Komiya (2001).

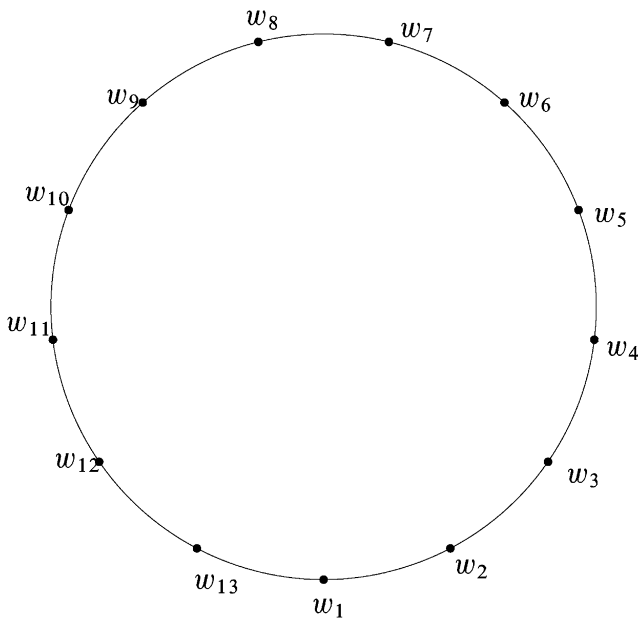

Fig. 3. S^1.

included in a small triangle which consists of (v_1, v_3), (v_2, v_3) and (v_2, v_4) we define the following function (see Fig. 2).

$$F(\alpha(v_1, v_3) + \beta(v_2, v_3) + \gamma(v_2, v_4)) = \frac{\alpha F(v_1, v_3) + \beta F(v_2, v_3) + \gamma F(v_2, v_4)}{|\alpha F(v_1, v_3) + \beta F(v_2, v_3) + \gamma F(v_2, v_4)|} \quad (1)$$

where $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, $0 \leqq \gamma \leqq 1$ and $\alpha + \beta + \gamma = 1$. Then, we can obtain a continuous function for this triangle. By similar ways this continuous function is extended to the entire $R \times R$, and we obtain a continuous function for all points in $R \times R$ from a discrete social welfare function on the vertices of $R \times R$. Denote this continuous function by $F : R \times R \to S^1$. Let us see that such a continuous function is well defined over $R \times R$. By IIA, for example, if $F(v_1, v_7) = w_1$, we must have $F(v_2, v_7) = w_1$ or w_2 or w_3. As this example shows, since individual preferences represented by adjacent two vertices of $R \times R$ are equal for two pairs of alternatives (except between v_{12} and v_{13}, and between v_{13} and v_1), if the preference of one of two individuals changes, the social preference moves at most two vertices clockwise or counter-clockwise along S^1, or moves three vertices only when the social preference moves across w_{13}. Thus, if the preferences of two individuals change, the social preference moves at most five vertices clockwise or counter-clockwise along S^1, and hence the social preference does not move across the antipodal point on S^1. Therefore, $(\alpha F(v_1, v_3) + \beta F(v_2, v_3) + \gamma F(v_2, v_4)/|\alpha F(v_1, v_3) + \beta F(v_2, v_3) + \gamma F(v_2, v_4)|)$ and so on are well defined. We will present some detailed discussion about the construction of a continuous function from a social welfare function for triangles including the vertices (v_{13}, v_{12}), (v_{13}, v_{13}) and so on in the later section (after the proof of Lemma 2).

Now we consider the following set Δ of the vertices of $R \times R$.

$$\Delta = \{(v_1, v_1), (v_2, v_2), (v_3, v_3), (v_4, v_4), (v_5, v_5), (v_6, v_6), (v_7, v_7), (v_8, v_8), (v_9, v_9), (v_{10}, v_{10}), (v_{11}, v_{11}), (v_{12}, v_{12}), (v_{13}, v_{13}), (v_1, v_1)\}$$

The diagram obtained by connecting these vertices is also denoted by Δ. It is homeomorphic to R. The profiles of two individuals when the preference of individual B is fixed at v_1, and the profiles when the preference of individual A is fixed at v_1 are denoted, respectively, by $A = \{(p_A, p_B) : p_B = v_1\}$ and $B = \{(p_A, p_B) : p_A = v_1\}$. The diagrams obtained by connecting vertices of A, and similarly obtained from B are also denoted, respectively, by A and B. They are also homeomorphic to R.

The union of these three sets $\Delta \cup A \cup B$ is depicted as the boundary ∂T_1 of the triangle T_1 in Fig. 4. $\Delta \cup A \cup B$ is homeomorphic to the circumference S^1. The vertices at four corners of the square depicted in Fig. 4 represent the same profile (v_1, v_1). The values of F for them are equal. The 1-dimensional homology group of $\Delta \cup A \cup B$ is also isomorphic to $\mathbb{Z}$, that is, $H_1(\Delta \cup A \cup B) \cong \mathbb{Z}$.

3. The Brouwer fixed point theorem

In this section we show the following theorem about homotopy and the degree of mapping of continuous functions on an $(n-1)$-dimensional sphere.

Note: Let F be a function from $(n-1)$-dimensional sphere S^{n-1} to itself, and F_* be the homomorphism of homology groups induced by F,

$$F_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$$

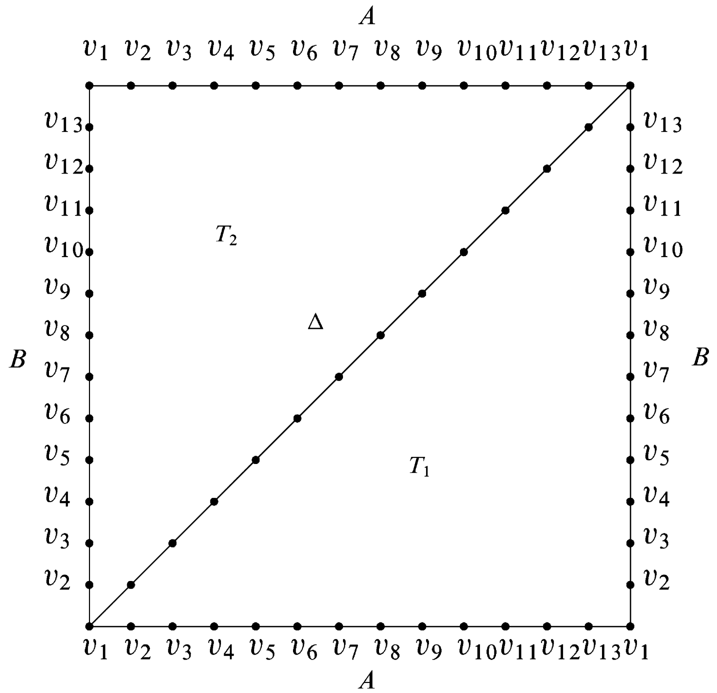

Fig. 4. $\Delta \cup A \cup B$.

$H_{n-1}(S^{n-1})$ is the $(n-1)$-dimensional homology group of S^{n-1}. Then, the degree of mapping of F, which is denoted by d_F, is defined as an integer which satisfies

$$F_*(h) = d_F h \text{ for } h \in H_{n-1}(S^{n-1})$$

Now we show

Theorem 1. *The following two results are equivalent.*

(1) *If there exists a continuous function from an n-dimensional ball D^n to an $(n-1)$-dimensional sphere $S^{n-1}(n \geqq 2)$, $F : D^n \to S^{n-1}$, then the following function, which is obtained by restricting F to the boundary S^{n-1} of D^n,*

$$F|_{S^{n-1}} : S^{n-1} \to S^{n-1}$$

is homotopic to a constant mapping. Since the degree of mapping of a constant mapping is zero, the degree of mapping of $F|_{S^{n-1}}$ is zero.

(2) *The Brouwer fixed point theorem: Any continuous function from D^n to D^n $(n \geqq 2)$, $G : D^n \to D^n$, has a fixed point.*

Proof.

(1) (1) → (2)

Assume that G has no fixed point. Since we always have $v \neq G(v)$ at any point v in D^n, there is a half line starting from $G(v)$ across v.[5] Let $F(v)$ be the intersection point of this half line and the boundary of D^n, which is S^{n-1}, then we obtain the following continuous function from D^n to S^{n-1}.

$$F : D^n \to S^{n-1}$$

In particular, we have $F(v) = v$ for $v \in S^{n-1}$. Therefore, $F|_{S^{n-1}}$ is an identity mapping. But, because an identity mapping on S^{n-1} is not homotopic to any constant mapping, it is a contradiction.

(2) (2) → (1)

[5] If v is a fixed point, $G(v)$ and v coincide, and hence there does not exist such a half line.

We show that if there exists a continuous function F from D^n to S^{n-1}, (1) of this theorem is correct whether a continuous function G from D^n to D^n has a fixed point or not. Define $f_t(v) = F[(1-t)v](0 \leqq t \leqq 1)$ for any point v of S^{n-1}. Then, we get a continuous function $f_t : S^{n-1} \to S^{n-1}$. $(1-t)v$ is a point which divide $t : 1-t$ a line segment between v and the center of D^n, and it is transferred by F to a point on S^{n-1}. Since f_t is continuous with respect to t, we have $f_0 = F|_{S^{n-1}}$, and $f_1 = F(0)$ is a constant mapping whose image is a point $F(0)$. Therefore, f_t is a homotopy from $F|_{S^{n-1}}$ to a constant mapping, and the degree of mapping of $F|_{S^{n-1}}$ is zero. □

An implication of this theorem is as follows.

Corollary 1. *Suppose that there exists a function from D^n to S^{n-1}, $F : D^n \to S^{n-1}$. If its restriction to S^{n-1}, $F|_{S^{n-1}} : S^{n-1} \to S^{n-1}$, is not homotopic to a constant mapping, F cannot be continuous.*

In relation to a social welfare function on $R \times R$, if there exists a function F defined on the vertices of $R \times R$, we can obtain a continuous function over $R \times R$ from F by the way explained above. Then, there exists a continuous function defined on T_1. Since T_1 is homeomorphic to D^2, and $\Delta \cup A \cup B$ is homeomorphic to S^1, the restriction of F to $\Delta \cup A \cup B$, $F|_{\Delta \cup A \cup B}$, must be homotopic to a constant mapping. If, when we require that the conditions of Pareto principle, IIA and non-existence of dictator must be satisfied by a social welfare function defined on the vertices $R \times R$, the restriction of this social welfare function to $\Delta \cup A \cup B$ is not homotopic to a constant mapping, then there does not exist such a social welfare function in the first place.

4. The main result

First we show the following lemma.

Lemma 1. *If neither individual* A *nor* B *is a dictator, all vertices of A and B correspond to w_1 in S^1 by F.*

Proof. As a first step we show that if individual A is not a dictator, the profile (v_{11}, v_1) must correspond to w_1. By Pareto principle (v_{11}, v_1) corresponds to w_1 or w_{11} or w_{12} in S^1. First assume that (v_{11}, v_1) corresponds to w_{12}. This means that when individual A prefers x_3 to x_2 and individual B prefers x_2 to x_3, the society is indifferent between them. Thus, by Pareto principle and IIA (v_9, v_1) must correspond to w_{12}. It implies that when individual A prefers x_3 to x_1 and individual B prefers x_1 to x_3, the society prefers x_1 to x_3. Similarly, (v_7, v_3) must correspond to w_6 because both individuals prefer x_2 to x_1, individual A prefers x_3 to x_2, and individual B prefers x_2 to x_3 at (v_7, v_3). But then, the society prefers x_3 to x_1 even when individual A prefers x_3 to x_1 and individual B prefers x_1 to x_3. It is a contradiction.

Next we show that if $(v_{11}, v_1) \to w_{11}$, individual A is the dictator. This correspondence means that when individual A prefers x_3 to x_2 and individual B prefers x_2 to x_3, the society prefers x_3 to x_2. Then, by Pareto principle and IIA we obtain

$$(v_7, v_3) \to w_7, (v_7, v_4) \to w_7$$

These mean that when individual A prefers x_3 to x_1 and individual B prefers x_1 to x_3 or is indifferent between them, the society prefers x_3 to x_1. Then, we get

$$(v_5, v_1) \to w_5, (v_5, v_2) \to w_5, (v_9, v_{12}) \to w_9$$

These mean that when individual A prefers x_2 to x_1 and individual B prefers x_1 to x_2 or is indifferent between them, the society prefers x_2 to x_1, and that when individual A prefers x_3 to x_2 and individual B is indifferent between them, the society prefers x_3 to x_2. Then, we get

$$(v_3, v_{11}) \to w_3, (v_3, v_{12}) \to w_3$$

These mean that when individual A prefers x_2 to x_3 and individual B prefers x_3 to x_2 or is indifferent between them, the society prefers x_2 to x_3. Then, we get

$$(v_1, v_9) \to w_1, (v_1, v_{10}) \to w_1$$

These mean that when individual A prefers x_1 to x_3 and individual B prefers x_3 to x_1 or is indifferent between them, the society prefers x_1 to x_3. Then, we get

$$(v_{11}, v_7) \to w_{11}, (v_{11}, v_8) \to w_{11}$$

These mean that when individual A prefers x_1 to x_2 and individual B prefers x_2 to x_1 or is indifferent between them, the society prefers x_1 to x_2. These correspondences mean that individual A is the dictator. Therefore, if individual A is not a dictator, we must have the correspondence $(v_{11}, v_1) \to w_1$. This correspondence means that when individual A prefers x_3 to x_2 and individual B prefers x_2 to x_3, the society prefers x_2 to x_3. By Pareto principle and IIA we obtain

$$(v_9, v_1) \to w_1, (v_{10}, v_1) \to w_1$$

These mean that when individual A prefers x_3 to x_1 or is indifferent between them and individual B prefers x_1 to x_3, the society prefers x_1 to x_3. Then, we get

$$(v_7, v_{11}) \to w_{11}, (v_8, v_{11}) \to w_{11}, (v_6, v_3) \to w_3$$

These mean that when individual A prefers x_2 to x_1 or is indifferent between them and individual B prefers x_1 to x_2, the society prefers x_1 to x_2, and that when individual A is indifferent between x_2 and x_3 and individual B prefers x_2 to x_3, the society prefers x_2 to x_3. Then, by Pareto principle and IIA we obtain the correspondences from profiles of individual preferences to the social preference when the preference of individual B is fixed at v_1 as follows,

$$\begin{aligned}
&(v_1, v_1) \to w_1, (v_2, v_1) \to w_1, (v_3, v_1) \to w_1\\
&(v_4, v_1) \to w_1, (v_5, v_1) \to w_1, (v_6, v_1) \to w_1\\
&(v_7, v_1) \to w_1, (v_8, v_1) \to w_1, (v_9, v_1) \to w_1\\
&(v_{10}, v_1) \to w_1, (v_{11}, v_1) \to w_1, (v_{12}, v_1) \to w_1\\
&(v_{13}, v_1) \to w_1
\end{aligned}$$

Interchanging the role of individual A and that of individual B, if individual B is not a dictator, the correspondences from profiles of individual preferences to the social preference when the preference of individual A is fixed at v_1 are obtained as follows,

$$\begin{aligned}
&(v_1, v_1) \to w_1, (v_1, v_2) \to w_1, (v_1, v_3) \to w_1\\
&(v_1, v_4) \to w_1, (v_1, v_5) \to w_1, (v_1, v_6) \to w_1\\
&(v_1, v_7) \to w_1, (v_1, v_8) \to w_1, (v_1, v_9) \to w_1 \qquad \square\\
&(v_1, v_{10}) \to w_1, (v_1, v_{11}) \to w_1, (v_1, v_{12}) \to w_1\\
&(v_1, v_{13}) \to w_1
\end{aligned}$$

Based on Lemma 1 we show the following result.

Lemma 2. *Suppose that there exists a social welfare function $F : R \times R \to S^1$ which satisfies Pareto principle and IIA. If F has no dictator, the degree of mapping of $F|_{\Delta \cup A \cup B}$ is not zero, and hence it is not homotopic to a constant mapping.*

Proof. By Pareto principle the following vertices of Δ correspond to the vertices of S^1 by F as follows.

$$\begin{aligned}
&(v_1, v_1) \to w_1, (v_3, v_3) \to w_3, (v_5, v_5) \to w_5\\
&(v_7, v_7) \to w_7, (v_9, v_9) \to w_9, (v_{11}, v_{11}) \to w_{11}
\end{aligned}$$

For other vertices of Δ, which represent profiles of two individuals such that they are indifferent about at least one pair of alternatives, we can not determine correspondences to the vertices of S^1 only by Pareto principle. By Pareto principle the correspondences from these vertices except for (v_{12}, v_{12}) and (v_{13}, v_{13}) to the social preference should be as follows,

$$\begin{aligned}
&(v_2, v_2) \to w_1, w_2 \text{ or } w_3, (v_4, v_4) \to w_3, w_4 \text{ or } w_5, (v_6, v_6) \to w_5, w_6 \text{ or } w_7\\
&\qquad (v_8, v_8) \to w_7 \text{ or } w_8 \text{ or } w_9, (v_{10}, v_{10}) \to w_9 \text{ or } w_{10} \text{ or } w_{11}
\end{aligned}$$

Let us consider the correspondence from (v_2, v_2). When (v_2, v_2) corresponds to w_2, the arc connecting (v_1, v_1), (v_2, v_2) and (v_3, v_3) in Δ corresponds to the arc connecting w_1, w_2 and w_3 in S^1. If (v_2, v_2) corresponds to w_1, we consider a barycentric subdivision of the 1-dimensional simplex $\langle\langle v_2, v_2), (v_3, v_3)\rangle\rangle$. Consider another vertex of Δ, $(v_{2,3}, v_{2,3})$, which is the middle point of the arc between (v_2, v_2) and (v_3, v_3). Then, $(v_{2,3}, v_{2,3})$ is transferred to w_2 by F, and the arc connecting (v_1, v_1), (v_2, v_2) and (v_3, v_3) through $(v_{2,3}, v_{2,3})$ corresponds to the arc connecting w_1, w_2 and w_3. If (v_2, v_2) corresponds to w_3, we consider another vertex of Δ, $(v_{1,2}, v_{1,2})$, which is the middle point of the arc between (v_1, v_1) and (v_2, v_2). Then, $(v_{1,2}, v_{1,2})$ is transferred to w_2 by F, and the arc connecting (v_1, v_1), (v_2, v_2) and (v_3, v_3) through $(v_{1,2}, v_{1,2})$ corresponds to the arc connecting w_1, w_2 and w_3. By similar methods we obtain the following results.

The arc connecting (v_3, v_3), (v_4, v_4) and (v_5, v_5) corresponds to the arc connecting w_3, w_4 and w_5
The arc connecting (v_5, v_5), (v_6, v_6) and (v_7, v_7) corresponds to the arc connecting w_5, w_6 and w_7
The arc connecting (v_7, v_7), (v_8, v_8) and (v_9, v_9) corresponds to the arc connecting w_7, w_8 and w_9

and

The arc connecting (v_9, v_9), (v_{10}, v_{10}) and (v_{11}, v_{11}) corresponds to the arc connecting w_9, w_{10} and w_{11}

About the correspondences from (v_{13}, v_{13}) there are the following three cases.

(1) Case 1: (v_{13}, v_{13}), which represents a profile such that two individuals are indifferent about x_1, x_2 and x_3, corresponds to a strict social preference. Assume that (v_{13}, v_{13}) corresponds to w_1. Then, by Pareto principle and IIA (v_{12}, v_{12}) is

transferred to w_1 by F. We can proceed the arguments in a similar manner based on other assumptions.[6] Let us consider a barycentric subdivision of the 1-dimensional simplex $\langle(v_{11}, v_{11}), (v_{12}, v_{12})\rangle$. Consider two new vertices of Δ, $(v^1_{11,12}, v^1_{11,12})$ and $(v^2_{11,12}, v^2_{11,12})$, which are two points that trisect the arc between (v_{11}, v_{11}) and (v_{12}, v_{12}). Then, $(v^1_{11,12}, v^1_{11,12})$ and $(v^2_{11,12}, v^2_{11,12})$ are transferred to, respectively, w_{12} and w_{13} by F, and the arc connecting $(v_{11}, v_{11}), (v_{12}, v_{12})$ and (v_{13}, v_{13}) through $(v^1_{11,12}, v^1_{11,12})$ and $(v^2_{11,12}, v^2_{11,12})$ corresponds to the arc connecting w_{11}, w_{12}, w_{13} and w_1.

(2) Case 2: (i) (v_{13}, v_{13}) corresponds to a social preference which is partially strict. Assume that (v_{13}, v_{13}) corresponds to w_{12}. Then, by Pareto principle and IIA (v_{12}, v_{12}) is transferred to w_{12} by F. We can proceed the arguments in a similar manner based on other assumptions.[7] Let us consider a barycentric subdivision of the 1-dimensional simplex $\langle(v_{13}, v_{13}), (v_1, v_1)\rangle$. Consider another vertex of Δ, $(v_{13,1}, v_{13,1})$, which is the middle point of the arc between (v_{13}, v_{13}) and (v_1, v_1). Then, $(v_{13,1}, v_{13,1})$ is transferred to w_{13} by F, and the arc connecting (v_{12}, v_{12}), (v_{13}, v_{13}) and (v_1, v_1) through $(v_{13,1}, v_{13,1})$ corresponds to the arc connecting w_{12}, w_{13} and w_1.[8]

(ii) Next assume that (v_{13}, v_{13}) corresponds to w_2. Then, by Pareto principle and IIA (v_{12}, v_{12}) is transferred to w_1 by F. We can proceed the arguments in a similar manner based on other assumptions.[9] Considering the same barycentric subdivision of the 1-dimensional simplex $\langle(v_{11}, v_{11}), (v_{12}, v_{12})\rangle$ as that in Case 1, we can show that the arc connecting $(v_{11}, v_{11}), (v_{12}, v_{12})$ through $(v^1_{11,12}, v^1_{11,12})$ and $(v^2_{11,12}, v^2_{11,12})$ corresponds to the arc connecting w_{11}, w_{12}, w_{13} and w_1.[10]

(3) Case 3: (v_{13}, v_{13}) corresponds to a social preference which is not strict. In this case (v_{13}, v_{13}) corresponds to w_{13}. By Pareto principle and IIA (v_{12}, v_{12}) is transferred to w_{12} by F. In this case we do not need to consider a barycentric subdivision.

A set of 1-dimensional simplices which is a 1-dimensional cycle of $\Delta \cup A \cup B$ is represented as follows.

$$\begin{aligned} z = {} & \langle(v_1, v_1), (v_2, v_1)\rangle + \langle(v_2, v_1), (v_3, v_1)\rangle + \langle(v_3, v_1), (v_4, v_1)\rangle + \cdots + \langle(v_{11}, v_1), (v_{12}, v_1)\rangle + \langle(v_{12}, v_1), (v_{13}, v_1)\rangle \\ & + \langle(v_{13}, v_1), (v_1, v_1)\rangle + \langle(v_1, v_1), (v_1, v_2)\rangle + \langle(v_1, v_2), (v_1, v_3)\rangle + \langle(v_1, v_3), (v_1, v_4)\rangle + \cdots + \langle(v_1, v_{11}), (v_1, v_{12})\rangle \\ & + \langle(v_1, v_{12}), (v_1, v_{13})\rangle + \langle(v_1, v_{13}), (v_1, v_1)\rangle + \langle(v_1, v_1), (v_{13}, v_{13})\rangle + \langle(v_{13}, v_{13}), (v_{12}, v_{12})\rangle + \langle(v_{12}, v_{12}), (v_{11}, v_{11})\rangle \\ & + \langle(v_{11}, v_{11}), (v_{10}, v_{10})\rangle + \cdots + \langle(v_3, v_3), (v_2, v_2)\rangle + \langle(v_2, v_2), (v_1, v_1)\rangle \end{aligned}$$

Since this simplicial complex has no 2-dimensional simplex, It is a representative element of a homology class of $\Delta \cup A \cup B$. z is transferred by F_*, which is the homomorphism of homology groups induced by F, to the following z'.

$$\begin{aligned} z' = {} & \langle w_1, w_{13}\rangle + \langle w_{13}, w_{12}\rangle + \langle w_{12}, w_{11}\rangle + \langle w_{11}, w_{10}\rangle + \langle w_{10}, w_9\rangle + \langle w_9, w_8\rangle + \langle w_8, w_7\rangle + \langle w_7, w_6\rangle + \langle w_6, w_5\rangle \\ & + \langle w_5, w_4\rangle + \langle w_4, w_3\rangle + \langle w_3, w_2\rangle + \langle w_2, w_1\rangle \end{aligned}$$

This is a cycle of S^1. Therefore, the homology group induced by $(F_*)|_{\Delta\cup A\cup B}$, which is the homomorphism of homology groups induced by $F|_{\Delta\cup A\cup B}$, is not trivial, and hence the degree of mapping of $F|_{\Delta\cup A\cup B}$ is not zero. □

Notes on the construction of a continuous function from a discrete social welfare function in each case of Lemma 2:

(1) Case 1: Consider a small triangle (in Fig. 2) which consists of $(v_{13}, v_{12}) = ((\overline{123}), (1\overline{23}))$, $(v_1, v_{12}) = ((123), (1\overline{23}))$ and $(v_1, v_{13}) = ((123), (\overline{123}))$. From above results (v_1, v_{12}) and (v_1, v_{13}) correspond to w_1, and the correspondence $(v_2, v_1) \rightarrow w_1$ implies that (v_{13}, v_{12}) also corresponds to w_1. Consider another small triangle which consists of (v_{13}, v_{12}), $(v_{13}, v_{13}) = ((\overline{123}), (\overline{123}))$ and (v_1, v_{13}). (v_{13}, v_{13}) corresponds to w_1. Thus, a continuous function derived from a discrete social welfare function described in (1) is well defined. Similar for other triangles.

(2) Case 2: (i) Consider a small triangle which consists of (v_{13}, v_{12}), (v_{13}, v_{13}) and (v_1, v_{13}). In this case (v_{13}, v_{13}) corresponds to w_{12}, (v_1, v_{13}) corresponds to w_1, and $(v_2, v_1) \rightarrow w_1$ implies that (v_{13}, v_{12}) corresponds to w_{12}. Thus, a continuous function derived from a discrete social welfare function described in (1) is well defined. Similar for other triangles.

(ii) Consider the same small triangle. In this case (v_{13}, v_{13}) corresponds to w_2, (v_1, v_{13}) corresponds to w_1 and (v_{13}, v_{12}) corresponds to w_1. Thus, a continuous function derived from a discrete social welfare function described in (1) is well defined. Similar for other triangles.

(3) Case 3: Consider a small triangle which consists of (v_{13}, v_{12}), (v_{13}, v_{13}) and (v_1, v_{13}). In this case (v_{13}, v_{13}) corresponds to w_{13}, (v_1, v_{13}) corresponds to w_1 and (v_{13}, v_{12}) corresponds to w_{12}. Thus, a continuous function derived from a discrete social welfare function described in (1) is well defined. Similar for other triangles.

From Theorem 1 and Lemma 2 we obtain the following result.

[6] For example, if (v_{13}, v_{13}) corresponds to w_9, interchanging x_1, x_2, x_3 by x_3, x_1, x_2 the arguments are essentially identical.

[7] For example, if (v_{13}, v_{13}) corresponds to w_4, interchanging x_1, x_2 by x_2, x_1 the arguments are essentially identical.

[8] In (i) of Case 2 (v_{13}, v_{13}) corresponds to a social preference such that the society is indifferent between two alternatives, and prefer the third alternative to these alternatives.

[9] For example, if (v_{13}, v_{13}) corresponds to w_{10}, interchanging x_2, x_3 by x_3, x_2 the arguments are essentially identical.

[10] In (ii) of Case 2 (v_{13}, v_{13}) corresponds to a social preference such that the society is indifferent between two alternatives, and prefer these alternatives to the third alternative.

Theorem 2. *The non-existence of social welfare function which satisfies Pareto principle, IIA and has no dictator (the Arrow impossibility theorem) is equivalent to the Brouwer fixed point theorem.*

5. Concluding remarks

We have shown that in the case of two individuals and three alternatives (or under the assumption of the free-triple property) the Arrow impossibility theorem that there exists no binary social choice rule which satisfies transitivity, Pareto principle, IIA and has no dictator is equivalent to the Brouwer fixed point theorem on a 2-dimensional ball (circle) using the elementary concepts and techniques of algebraic topology, in particular, homology groups of simplicial complexes, homomorphisms of homology groups induced by simplicial mappings.

Our approach may be applied to other social choice problems such as Wilson's impossibility theorem (Wilson, 1972), the Gibbard–Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) and Amartya Sen's liberal paradox (Sen, 1979).

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