Decisions Econ Finan (2009) 32:67–78
DOI 10.1007/s10203-009-0085-8

NOTES AND COMMENTS

On the computability of quasi-transitive binary social choice rules in an infinite society and the halting problem

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Published online: 7 March 2009

Abstract This paper investigates the computability problem of the existence of a vetoer and an oligarchy for quasi-transitive binary social choice rules (Mas-Colell and Sonnenschein in Rev Econ Stud 39:185–192, 1972) in a society with an infinite number of individuals (infinite society) according to the computable calculus (or computable analysis) by Aberth (Computable analysis, McGraw-Hill, New York, 1980; Computable calculus, Academic Press, Dublin, 2001). We will show the following results. The problem whether a quasi-transitive binary social choice rule which satisfies Pareto principle and independence of irrelevant alternatives (IIA) has a vetoer or has no vetoer in an infinite society is a nonsolvable problem, that is, there exists no ideal computer program for a quasi-transitive binary social choice rule which satisfies Pareto principle and IIA that decides whether it has a vetoer or has no vetoer. And it is equivalent to nonsolvability of the halting problem. We also show that if for any quasi-transitive binary social choice rule there exists an oligarchy in an infinite society, whether it is finite or infinite is a nonsolvable problem. A vetoer is an individual such that if he strictly prefers an alternative to another alternative, then the society prefers the former to the latter or is indifferent between them regardless of the preferences of other individuals, and an oligarchy is the minimal set of individuals which has dictatorial power and its each member is a vetoer. It will be shown that an oligarchy is a set of vetoers if it exists.

This research was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (C), 20530165, and the Grant-in-Aid from the Zengin Foundation for Studies on Economics and Finance in Japan.

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Keywords Quasi-transitive social choice rule · Vetoer · Oligarchy · Infinite society · Nonsolvable problem · Halting problem

1 Introduction

This paper investigates the computability problem of the existence of a vetoer and an oligarchy for quasi-transitive binary social choice rules (Mas-Colell and Sonnenschein 1972) in a society with an infinite number of individuals (infinite society) according to the computable calculus (or computable analysis) by Aberth (1980, 2001).[1] Arrow's impossibility theorem (Arrow 1963) shows that, with a finite number of individuals, for any social welfare function (transitive binary social choice rule) which satisfies Pareto principle and independence of irrelevant alternatives (IIA) there exists a dictator. In contrast Fishburn (1970), Hansson (1976) and Kirman and Sondermann (1972) show that in a society with an infinite number of individuals (infinite society) there exists a social welfare function without dictator. Hansson (1976) also shows that in an infinite society there exists a quasi-transitive binary social choice rule which satisfies Pareto principle and IIA without dictator.[2] We abbreviate quasi-transitive binary social choice rule as Q-BCR. Quasi-transitivity requires only transitivity of strict social preferences. On the other hand the ordinary transitivity (or full transitivity) requires transitivity of preference-indifference relations and indifference relations as well as transitivity of strict preference relations. Mas-Colell and Sonnenschein (1972) shows that, with a finite number of individuals, there exists a vetoer for any Q-BCR. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society strictly prefers the former to the latter regardless of the preferences of other individuals, and a vetoer is an individual such that if he strictly prefers an alternative to another alternative, then the society prefers the former to the latter or is indifferent between them regardless of the preferences of other individuals.

In the next section, we present the framework of this paper and some preliminary results. In Sect. 3 we will show the following results. The problem whether a quasi-transitive binary social choice rule which satisfies Pareto principle and IIA has a vetoer or has no vetoer in an infinite society is a nonsolvable problem, that is, there exists no ideal computer program for a quasi-transitive binary social choice rule which satisfies Pareto principle and IIA that decides whether it has a vetoer or has no vetoer. And it is equivalent to nonsolvability of the halting problem. We also show that if there exists an oligarchy for such a quasi-transitive binary social choice rule in an infinite society, whether it is finite or infinite is also a nonsolvable problem. An oligarchy is the minimal set of individuals which has dictatorial power and its each member is a vetoer.

[1] In another paper (Tanaka 2008), we studied the computability problem of Arrovian binary social choice rules in an infinite society. We showed that the problem whether a transitive binary social choice rule satisfying Pareto principle and independence of irrelevant alternatives (IIA) has a dictator or has no dictator in an infinite society is a nonsolvable problem, that is, there exists no ideal computer program for a transitive binary social choice rule satisfying Pareto principle and IIA that decides whether the binary social choice rule has a dictator or has no dictator. And it is equivalent to nonsolvability of the halting problem.

[2] Taylor (2005) is a recent book that discusses social choice problems in an infinite society.

2 The framework and preliminary results

There are more than two (finite or infinite) alternatives and a countably infinite number of individuals. The set of individuals is denoted by ω, and the set of alternatives is denoted by A. The individuals are appropriately enumerated. The alternatives are represented by x, y, z, w and so on. Individual preferences over the alternatives are transitive weak orders, that is, individuals prefer one alternative to another alternative, or are indifferent between them. Denote individual i's preference by $\succ_i$. We denote $x \succ_i y$ when individual i prefers x to y, and denote $x \sim_i y$ when he is indifferent between x and y. A combination of individual preferences, which is called a *profile*, is denoted by $p(= (\succ_1, \succ_2, \ldots))$, $p'(= (\succ'_1, \succ'_2, \ldots))$ and so on. We assume that the profiles satisfy the free triple property. It means that about any set of three alternatives, the profiles of individual preferences are not restricted. About a set of three alternative (denoted by $\{x, y, z\}$) we denote the set of preferences of individual i by Σ^i_{xyz}. The set of profiles about $\{x, y, z\}$ is denoted by Σ^{ω}_{xyz}, where $\omega = \{1, 2, \ldots\}$ is the set of natural numbers. It represents the set of individuals.

We consider a binary social choice rule which determines a social preference corresponding to each profile. It is abbreviated as a Q-BCR. Social preferences are defined similarly to individual preferences. But we require only quasi-transitivity for social preferences, that is, they must satisfy only transitivity of strict preferences. We denote $x \succ y$ when the society strictly prefers x to y, and denote $x \sim y$ when it is indifferent between x and y. The social preferences is denoted by $\succ$ at p, by $\succ'$ at p' and so on.

Ideal computer Now we consider an ideal computer according to Aberth (2001). An ideal computer is a machine that manipulates symbol strings, and these symbol strings may be arbitrarily long. The ideal computer has a finite number of registers. Initially all registers are empty of symbol strings, except for a few registers, $v_1, v_2, \ldots, v_n$, this being the inputs to the ideal computer. The outputs of the ideal computer, after it ceases computation, is the contents of another group of registers, $w_1, w_2, \ldots, w_m$. If P is the program of the ideal computer, with its registers $v_1, v_2, \ldots, v_n$ set to prescribed values $a_1, a_2, \ldots, a_n$, respectively, then $P(a_1, a_2, \ldots, a_n)$ designates its outputs after computation terminates, that is, the values that leave in $w_1, w_2, \ldots, w_m$. An ideal computer for a social choice rule will be explained in the next section.

Social preferences are further required to satisfy *Pareto principle* and *independence of irrelevant alternatives (IIA)*. The meanings of these conditions are as follows:

Pareto principle When all individuals prefer an alternative x to another alternative y, the society must prefer x to y.

Independence of irrelevant alternatives The social preference about every pair of two alternatives x and y is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x and y.

Mas-Colell and Sonnenschein (1972) shows that, with a finite number of individuals, for any Q-BCR there exists a vetoer. According to definitions in Sen Amartya (1979) we define the following terms.

Almost decisiveness If, when all individuals in a (finite or infinite) group G prefer an alternative x to another alternative y, and other individuals (individuals in $\omega \backslash G$) prefer y to x, the society prefers x to y ($x \succ y$), then G is *almost decisive* for x against y.
Decisiveness If, when all individuals in a group G prefer x to y, the society prefers x to y regardless of the preferences of other individuals, then G is *decisive* for x against y.
Decisive set If a group of individuals is decisive about every pair of alternatives, it is called a decisive set.
Almost semi-decisiveness If, when all individuals in a group G prefer an alternative x to another alternative y, and other individuals (individuals in $\omega \backslash G$) prefer y to x, the society prefers x to y or is indifferent between them ($x \succsim y$), then G is *almost semi-decisive* for x against y.
Semi-decisiveness If, when all individuals in a group G prefer x to y, the society prefers x to y or is indifferent between them regardless of the preferences of other individuals, then G is *semi-decisive* for x against y.

A decisive set may consist of one individual. If an individual is decisive about every pair of alternatives for a Q-BCR, then he is a *dictator* of the Q-BCR. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter. Of course, there exists at most one dictator. On the other hand, if an individual is semi-decisive about every pair of alternatives, then he is a vetoer. There may exist more than one vetoers.

Next we define an *oligarchy*.
Oligarchy An oligarchy is the minimal decisive set such that its each member is a vetoer. *Minimal* means the following property.

> Let O be an oligarchy and i be a member of O. Then, $O \backslash \{i\}$ is not a decisive set for all $i \in O$.

First about decisiveness we can show the following lemma.

Lemma 1 *If a group of individuals G is almost decisive for an alternative x against another alternative y, then it is decisive about every pair of alternatives, that is, it is a decisive set.*

Proof See Appendix. □

This lemma is a standard result of social choice theory. But for convenience of readers we present its proof in the Appendix.

The implications of this lemma are similar to those of Lemma 3*a in Sen Amartya (1979) and Dictator Lemma in Suzumura (2000). Next we show the following lemma.

Lemma 2 *If G_1 and G_2 are decisive sets, then $G_1 \cap G_2$ is also a decisive set.*

Proof Let x, y and z be given three alternatives, and consider the following profile.

1. Individuals in $G_1 \backslash G_2$ (denoted by i): $z \succ_i x \succ_i y$
2. Individuals in $G_2 \backslash G_1$ (denoted by j): $y \succ_j z \succ_j x$
3. Individuals in $G_1 \cap G_2$ (denoted by k): $x \succ_k y \succ_k z$

4. Other individuals (denoted by l): $z \succ_l y \succ_l x$

Since G_1 and G_2 are decisive sets, the social preference is $x \succ y$ and $y \succ z$. Then, by quasi-transitivity the social preference about x and z should be $x \succ z$. Only individuals in $G_1 \cap G_2$ prefer x to z, and all other individuals prefer z to x. Thus, $G_1 \cap G_2$ is almost decisive for x against z. Then, by Lemma 1 it is a decisive set. □

Note that G_1 and G_2 can not be disjoint. Assume that G_1 and G_2 are disjoint. If individuals in G_1 prefer x to y, and individuals in G_2 prefer y to x, then neither G_1 nor G_2 can be a decisive set.

On the other hand, about semi-decisiveness we can show the following result.

Lemma 3 *If an individual* (*denoted by* i) *is almost semi-decisive for an alternative* x *against another alternative* y*, then he is semi-decisive about every pair of alternatives, that is, he is a vetoer.*

Proof 1. *Case 1*: There are more than three alternatives.
Let z and w be alternatives other than x and y, and consider the following profile.
   (a) Individual i: $z \succ_i x \succ_i y \succ_i w$.
   (b) Other individuals (denoted by j): $y \succ_j x$, $z \succ_j x$ and $y \succ_j w$. Their preferences about z and w are not specified.

   By Pareto principle the social preference is $z \succ x$ and $y \succ w$. Since i is almost semi-decisive for x against y, the social preference is $x \succsim y$. Assume that the social preference about z and w is $w \succ z$. Then, by quasi-transitivity the social preference should be $y \succ x$. It is a contradiction. Thus, we have $z \succsim w$. This means that i is semi-decisive for z against w. From this result we can show that i is semi-decisive for x (or y) against w, for z against x (or y), for y against x, and for x against y. Since z and w are arbitrary, i is semi-decisive about every pair of alternatives, that is, he is a vetoer.

2. *Case 2*: There are only three alternatives x, y and z.
Consider the following profile.
   (a) Individual i: $x \succ_i y \succ_i z$.
   (b) Other individuals (denoted by j): $y \succ_j z$, $y \succ_j x$, and their preferences about x and z are not specified.

   By Pareto principle the social preference is $y \succ z$. Since i is almost semi-decisive for x against y, the social preference is $x \succsim y$. Assume that the social preference about x and z is $z \succ x$. Then, by quasi-transitivity the social preference should be $y \succ x$. It is a contradiction. Thus, we have $x \succsim z$. This means that i is semi-decisive for x against z. Similarly we can show that i is semi-decisive for z against y considering the following profile.
   (a) Individual i: $z \succ_i x \succ_i y$.
   (b) Other individuals (denoted by j): $z \succ_j x$, $y \succ_j x$, and their preferences about y and z are not specified.

   By similar procedures we can show that i is semi-decisive for y against z, for z against x, for y against x, and for x against y. □

Further we show the following result.

Lemma 4 *The set of all vetoers is an oligarchy.*

Proof By the definition of oligarchy all vetoers are included in an oligarchy because if there exists a vetoer outside the oligarchy and he prefers x to y, then the oligarchy can not be decisive for y against x.

Next assume that an oligarchy includes an individual who is not a vetoer. Let denote him by i and the oligarchy by O, and assume the following profile

1. Individual i: $x \succ_i y$
2. Individuals in $O \backslash \{i\}$ (denoted by j): $y \succ_j x$
3. Other individuals (denoted by k): $y \succ_k x$

Since i is not a vetoer, the social preference about x and y is $y \succ x$.[3] Then, by Lemma 1 $\omega \backslash \{i\}$ is a decisive set. By Lemma 2 the intersection of O and $\omega \backslash \{i\}$ is also a decisive set. It is $O \backslash \{i\}$. But O is the minimal decisive set. Therefore, an oligarchy does not include an individual who is not a vetoer, and the set of all vetoers is an oligarchy. □

If there exists at least one vetoer and at least one individual who is not a vetoer, then we have an oligarchy which is a proper subset of ω. An oligarchy if it exists, may consist of a finite number of individuals or an infinite number of individuals. A dictator is an oligarchy of one individual. On the other hand, since by Pareto principle ω is a decisive set, if all individuals are vetoers, ω is the oligarchy. The following social choice rule is an example for which ω is the oligarchy.

Pareto extension rule

1. About any pair of alternatives (x, y), when all individuals prefer x to y, the society prefers x to y.
2. About any pair of alternatives (x, y), when some individuals prefer x to y, and all other individuals are indifferent between them, the society prefers x to y.
3. About any pair of alternatives (x, y), when some individuals prefer x to y, and some individuals prefer y to x, the society is indifferent between them.
4. About any pair of alternatives (x, y), when all individuals are indifferent between x and y, the society is indifferent between them.

We can construct a quasi-transitive social choice rule for which a proper subset of ω, which may be finite or infinite, is the oligarchy by modifying this Pareto extension rule. Consider the following quasi-transitive social choice rule.

1. About any pair of alternatives (x, y), when all individuals in a (finite or infinite) group G prefer x to y, the society prefers x to y regardless of the preferences of all other individuals.
2. About any pair of alternatives (x, y), when some individuals in G prefer x to y, and all other individuals in G are indifferent between them, the society prefers x to y regardless of the preferences of all other individuals.
3. About any pair of alternatives (x, y), when some individuals in G prefer x to y, and some individuals in G prefer y to x, the society is indifferent between them regardless of the preferences of all other individuals.
4. About any pair of alternatives (x, y), when all individuals in G are indifferent between x and y, the society is indifferent between them regardless of the preferences of all other individuals.

G is the oligarchy for this quasi-transitive social choice rule.

[3] If $x \succsim y$ individual i is a vetoer by Lemma 3.

3 The existence of vetoer and oligarchy, and the halting problem

Consider profiles such that about three alternatives x, y and z one individual (denoted by i) prefers x to y to z, and all other individuals prefer z to x to y. Denote such a profile by p^i, and denote the set of such profiles by $\bar{\Sigma}^{\omega}_{x,y,z}$. By Pareto principle the social preference about x and y is $x \succ y$. The social preference is "$x \succ z$" or "$z \succ y$" or "$x \sim z$ and $y \sim z$".[4] If the social preference is $x \succ z$ at p^i for some i, then by IIA individual i is almost decisive for x against y, and by Lemma 1 he is a dictator. Of course a dictator is a vetoer. If the social preference is $x \sim z$ and $y \sim z$ at p^i, by IIA individual i is almost semi-decisive for x against z, and by Lemma 3 he is a vetoer. On the other hand, if the social preference is $z \succ y$ at p^i for all $i \in N$, then there exists no vetoer. By IIA, a group $\omega \backslash \{i\}$ is almost decisive for z against y for all $i \in N$, and by Lemmas 1 and 2 all co-finite sets (sets of individuals whose complements are finite sets) are decisive sets. If $x \sim z$ and $y \sim z$ for only one i and $z \succ y$ for all other i, there exists no dictator. But the following lemma shows that it is impossible.

Lemma 5 *If there exists a unique vetoer, he is a dictator.*

Proof Let individual i be the unique vetoer and consider the following profile.

1. Individual i: $x \succ_i y \succ_i z$
2. An individual other than i (denoted by j): $z \succ_j x \succ_j y$
3. Other individuals (denoted by k): $x \succ_k z \succ_k y$

By Pareto principle the social preference is $x \succ y$. The social preference is "$x \succ z$" or "$z \succ y$" or "$x \sim z$ and $y \sim z$". If $z \succ y$, the set of individuals $\omega \backslash \{i\}$ is almost decisive for z against y, and by Lemma 1 it is a decisive set. Since individual i is a vetoer, it is impossible. If $x \sim z$, individual j is almost semi-decisive for z against x, and by Lemma 3 he is a vetoer. It is also impossible. Thus we have $x \succ z$. Then, the set of individuals $\omega \backslash \{j\}$ is almost decisive for x against z, and by Lemma 1 it is a decisive set. Since j is arbitrary, $\omega \backslash \{j\}$ is decisive for all $j \in N \backslash \{i\}$. Then, by Lemma 2 individual i is a dictator. □

From these arguments we obtain the following theorem.

Theorem 1 *Any Q-BCR has a vetoer or has no vetoer, and in the latter case all co-finite sets are decisive sets.*

We can show, however, that for any quasi-transitive binary social choice rule satisfying Pareto principle and IIA, the problem whether it has a vetoer or has no vetoer is a nonsolvable problem, that is, there exists no ideal computer program for a quasi-transitive binary social choice rule satisfying Pareto principle and IIA that decides whether it has a vetoer or has no vetoer.

Ideal computer for binary social choice rules We consider a program P of an ideal computer for such a quasi-transitive binary social choice rule restricted to profiles in

[4] If $z \succ x$, then $x \succ y$ and quasi-transitivity implies $z \succ y$. If $y \succ z$, then $x \succ y$ and quasi-transitivity implies $x \succ z$.

$\bar{\Sigma}^{\omega}_{x,y,z}$. The input I of P is a string of individual preferences $(\succ_1, \succ_2, \ldots)$. Possible preferences of each individual about x, y and z and also possible social preferences about x, y and z are, respectively, appropriately enumerated. The ideal computer reads the preference of each individual at the profile $p^i,\ i = 1, 2, \ldots,$ step by step from the preference of individual 1 at p^1, and registers them in sequence in the register v_1. It decides the social preference at $p^i,\ i = 1, 2, \ldots,$ after reading preferences of the first some individuals including individual i, that is, it decides the social preference at p^1 after reading preferences of individuals including individual 1, decides the social preference at p^2 after reading preferences of individuals including individual 2, and so on. And it registers the social preference at each profile in sequence in the register v_2.

If the social preference at p^1 is $x \succ z$ or "$x \sim z$ and $y \sim z$", then the ideal computer finds that individual 1 is a vetoer, writes "1" in the register w_1 whose value is its output, and it terminates; on the other hand if the social preference at p^1 is $z \succ y$, then the ideal computer does not find a vetoer and it continues to read the preference of individual 1 at p^2 in the next step. If the social preference at p^2 is $x \succ z$ or "$x \sim z$ and $y \sim z$", then it finds that individual 2 is a vetoer, writes "2" in w_1, and it terminates; on the other hand if the social preference at p^2 is $z \succ y$, then it does not find a vetoer and it continues to read the preference of individual 1 at p^3 in the next step, and so on. If the binary social choice rule has a vetoer, the ideal computer eventually finds a vetoer and terminates. On the other hand if the binary social choice rule does not have a vetoer, the ideal computer can not find a vetoer and it continues computation forever.

We show the following theorem which is the main result of this paper.

Theorem 2 1. *For any quasi-transitive binary social choice rule satisfying Pareto principle and IIA the problem whether the binary social choice rule has a vetoer or has no vetoer is a nonsolvable problem, that is, there exists no ideal computer program for any quasi-transitive binary social choice rule satisfying Pareto principle and IIA that decides whether it has a vetoer or has no vetoer.*
2. *The above result is equivalent to nonsolvability of the halting problem.*

Proof 1. We assume that there is an ideal computer program P^* which solves the problem whether the ideal computer program P for a quasi-transitive binary social choice rule finds a vetoer or not, that is, it terminates or not. The inputs to the program P^* are a program P in its register v_1 and a string of individual preferences I, which is the input to P, in v_2. P^* analyzes the program P with the input I, and supplies in w_1 a single output integer having two values, 1 to indicate that P finds a vetoer, and 0 to indicate that P does not find a vetoer. The 0–1 output of P^* is a function of P and I, and then we denote $P^*(P, I)$.

Next we define a program $P'(I)$ such that $P^*(P', I)$ is wrong. First, we construct another program P_S, whose inputs are two programs P^*, P and an integer K. In this formulation K denotes the maximum number of profiles P has read. Thus, we assume that P reads individual preferences until it decides the social preferences at $p^i,\ i = 1, 2, \ldots, K$, or P^* terminates before then. The program $P_S(P^*, P(I), K)$ follows the actions of $P^*(P, I)$ step by step. Then, P_S supplies three output integers. The first output integer is 0 if $P^*(P, I)$ does not terminate after P decides the social preference at p^K, and is 1 if $P^*(P, I)$ terminates just when P decides the social preference at p^K or before then. If the first output integer is 1, the remaining two output integers are significant, one giving the exact number of K, denoted by K^*, taken by $P^*(P, I)$ to termination, and the other giving the $P^*(P, I)$ output integer, 1 or 0, left in w_1 (of P^*).

The program $P'(I)$ employs P_S as a subroutine and behaves as follows.

(a) If P_S signals termination of $P^*(P', I)$ with the output 1 in w_1 (existence of vetoer), then $P'(I)$ gives the result that the social preference about y and z is $z \succ y$ at $p^i,\ i = 1, 2, \ldots$.

(b) If P_S signals termination of $P^*(P', I)$ with the output 0 in w_1 (non-existence of vetoer), then $P'(I)$ gives the result that the social preference about x and z is $x \succ z$ or "$x \sim z$ and $y \sim z$" at p^{K^*}.

(c) If P_S signals nontermination of $P^*(P', I)$ after P decides the social preference at p^K, then $P'(I)$ gives the result that the social preference about y and z is $z \succ y$ at $p^i,\ i = 1, 2, \ldots, K$.

Thus the binary social choice rule has a vetoer or has no vetoer, depending on whether P^* claims that it has no vetoer or has a vetoer, respectively. Whatever result P^* determines for P', the program P^* is wrong. And if P^* never terminate, it is still wrong because it fails to give a valid result that the quasi-transitive binary social choice rule has no vetoer.[5]

2. According to Aberth (2001) the halting problem is stated as follows.

The halting problem Let P be any program that receives its input I in a single register v_1, and P^* be a program with its inputs P in a register v_1 and I in v_2, and supplies in w_1 a single output integer, 1 to indicate termination for P and 0 to indicate nontermination for P. The halting problem is: Is there a program P^* that can determine whether P with that input will terminate or not terminate?

From the arguments before this theorem and the proof of the first part of this theorem it is clear that nonsolvability of the problem whether any quasi-transitive binary social choice rule satisfying Pareto principle and IIA has a vetoer or has no vetoer is equivalent to nonsolvability of the halting problem. □

Nonsolvability of the problem whether an oligarchy is infinite or finite if it exists

As mentioned in the previous section an oligarchy may be infinite or finite if it exists. About this result we can show the following theorem.

Theorem 3 *The problem whether an oligarchy is infinite or finite if it exists is a nonsolvable problem.*

Proof Let us assume that there exists at least one vetoer, and individual k be one of vetoers. Then we obtain the following results.

[5] This proof is based on the proof of nonsolvability of the problem to decide whether any real number equals zero or not in Aberth (2001).

1. If there exists a vetoer after individual k for some k, that is, the social preference is $x \sim z$ and $y \sim z$ at p^i for some i after k for some k,[6] then the oligarchy may be infinite.
2. On the other hand, if there exists no vetoer after individual k for some k, that is, the social preference is $z \succ y$ at p^i for all i after k for some k, then the oligarchy is finite.

We can show, however, that there exists no ideal computer program for a quasi-transitive binary social choice rule which satisfies Pareto principle and IIA that decides whether there exists a vetoer after individual k or not.

Consider a program P of an ideal computer for a quasi-transitive binary social choice rule restricted to profiles in $\bar{\Sigma}^{\omega}_{x,y,z}$. The input I of P is a string of individual preferences. The ideal computer performs similarly to that in Theorem 2. It reads the preference of each individual at the profile p^i, $i = k+1, k+2, \ldots$, step by step from the preference of individual 1 at p^{k+1}, and registers them in v_1. It decides the social preference at p^i, $i = k+1, k+2, \ldots$, after reading preferences of the first some individuals including individual i, that is, it decides the social preference at p^{k+1} after reading preferences of individuals including individual $k+1$, decides the social preference at p^{k+2} after reading preferences of individuals including individual $k+2$, and so on. And it registers the social preference at each profile in v_2.

If the social preference at p^{k+1} is "$x \sim z$ and $y \sim z$", individual $k+1$ is a vetoer, that is, there exists a vetoer after k, and the ideal computer writes "1" in w_1 whose value is its output, and terminates. On the other hand if the social preference at p^{k+1} is $z \succ y$, individual $k+1$ is not a vetoer, and the ideal computer continues to read the preference of individual 1 at p^{k+2} in the next step. If the social preference at p^{k+2} is "$x \sim z$ and $y \sim z$", individual $k+2$ is a vetoer, that is, there exists a vetoer after k, and the ideal computer writes "1" in w_1, and terminates. On the other hand if the social preference at p^{k+2} is $z \succ y$, individual $k+2$ is not a vetoer, and the ideal computer continues to read the preference of individual 1 at p^{k+3} in the next step, and so on. If there exists a vetoer after individual k, the ideal computer eventually finds a vetoer and terminates. On the other hand if there exists no vetoer after individual k, the ideal computer can not find a vetoer and it continues computation forever.

We assume that there is an ideal computer program P^* which solves the problem whether the ideal computer program P for a quasi-transitive binary social choice rule finds a vetoer after individual k or not, that is, it terminates or not. The inputs to the program P^* are a program P in its register v_1 and a string of individual preferences I, which is the input to P, in v_2. P^* supplies in w_1 a single output integer having two values, 1 to indicate that P finds a vetoer after individual k, and 0 to indicate that P does not find a vetoer after individual k. We denote $P^*(P, I)$.

Next we define a program $P'(I)$ such that $P^*(P', I)$ is wrong. First, we construct another program P_S, whose inputs are two programs P^*, P and an integer K. K denotes the maximum number of profiles P has read. Thus, we assume that P reads individual preferences until it decides the social preferences at p^i, $i = k+1, k+2, \ldots, K$, or P^* terminates before then. The program $P_S(P^*, P(I), K)$ follows the actions of

6 If the social preference is $x \succ z$ at some p^i, $i > k$, then i is a dictator. It contradicts the assumption that individual k is a vetoer.

$P^*(P, I)$ step by step. Then, P_S supplies three output integers. The first output integer is 0 if $P^*(P, I)$ does not terminate after P decides the social preference at p^K, and is 1 if $P^*(P, I)$ terminates just when P decides the social preference at p^K or before then. If the first output integer is 1, one of the remaining two output integers giving the exact number of K, denoted by K^*, taken by $P^*(P, I)$ to termination, and the other giving the $P^*(P, I)$ output integer, 1 or 0, left in w_1 (of P^*).

The program $P'(I)$ employs P_S as a subroutine and behaves as follows.

1. If P_S signals termination of $P^*(P', I)$ with the output 1 in w_1 (existence of vetoer after k), then $P'(I)$ gives the result that the social preference about y and z is $z \succ y$ at $p^i,\ i = k+1, k+2, \ldots$
2. If P_S signals termination of $P^*(P', I)$ with the output 0 in w_1 (non-existence of vetoer after k), then $P'(I)$ gives the result that the social preference about x and z is $x \succ z$ or "$x \sim z$ and $y \sim z$" at p^{K^*}.
3. If P_S signals nontermination of $P^*(P', I)$ after P decides the social preference at p^K, then $P'(I)$ gives the result that the social preference about y and z is $z \succ y$ at $p^i,\ i = k+1, k+2, \ldots, K$.

Thus the binary social choice rule has a vetoer or has no vetoer after k, depending on whether P^* claims that it has no vetoer or has a vetoer after k, respectively. Whatever result P^* determines for P', the program P^* is wrong. And if P^* never terminate, it is still wrong because it fails to give a valid result that the quasi-transitive binary social choice rule has no vetoer after k.

As proved in these arguments the problem whether there exists a vetoer after individual k or not is nonsolvable and is equivalent to the halting problem. Therefore, the problem whether the oligarchy is finite or infinite if it exists is also nonsolvable. □

4 Final remark

We have examined the existence of a vetoer or a dictator for quasi-transitive binary social choice rules in an infinite society. The assumption of an infinite society seems to be unrealistic. But Mihara (1997) presented an interpretation of an infinite society based on a *finite* number of individuals and a countably infinite number of uncertain states.

Appendix

A Proof of Lemma 1

1. *Case 1*: There are more than three alternatives.
   Let z and w be alternatives other than x and y, and consider the following profile.
   (a) Individuals in G (denoted by i): $z \succ_i x \succ_i y \succ_i w$.
   (b) Other individuals (denoted by j): $y \succ_j x$, $z \succ_j x$ and $y \succ_j w$. Their preferences about z and w are not specified.
   By Pareto principle the social preference is $z \succ x$ and $y \succ w$. Since G is almost decisive for x against y, the social preference is $x \succ y$. Then, by quasi-transitivity

the social preference should be $z \succ w$. This means that G is decisive for z against w. From this result we can show that G is decisive for x (or y) against w, for z against x (or y), for y against x, and for x against y. Since z and w are arbitrary, G is decisive about every pair of alternatives, that is, it is a decisive set.

2. *Case 2*: There are only three alternatives x, y and z.
   Consider the following profile.
   (a) Individuals in G (denoted by i): $x \succ_i y \succ_i z$.
   (b) Other individuals (denoted by j): $y \succ_j z$, $y \succ_j x$, and their preferences about x and z are not specified.

   By Pareto principle the social preference is $y \succ z$. Since G is almost decisive for x against y, the social preference is $x \succ y$. Then, by quasi-transitivity the social preference should be $x \succ z$. This means that G is decisive for x against z. Similarly we can show that G is decisive for z against y considering the following profile.
   (a) Individuals in G (denoted by i): $z \succ_i x \succ_i y$.
   (b) Other individuals (denoted by j): $z \succ_j x$, $y \succ_j x$, and their preferences about y and z are not specified.

   By similar procedures we can show that G is decisive for y against z, for z against x, for y against x, and for x against y. □

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