International Journal of Industrial Organization
19 (2001) 1423–1440

International Journal of Industrial Organization

www.elsevier.com/locate/econbase

Evolution to equilibrium in an asymmetric oligopoly with differentiated goods

Yasuhito Tanaka*

Faculty of Law, Chuo University, 742-1, Higashinakano, Hachioji, Tokyo, 192-0393, Japan

Received 24 November 1998; received in revised form 8 November 1999; accepted 21 January 2000

Abstract

We present results on finite population evolutionarily stable strategies (ESSs) and stochastically stable states for a model of evolution with an imitative rule of strategy choice in an asymmetric oligopoly with differentiated goods and two groups of firms, one group of low cost firms and one of high cost firms. The firms play price setting and quantity setting oligopoly games under linear demand functions. We will show that the stochastically stable state in a price setting oligopoly and that in a quantity setting oligopoly coincide.

Keywords: Asymmetric oligopoly with differentiated goods, Finite population ESS, Stochastically stable state

1. Introduction

Oligopolistic markets are typically analyzed under two alternative assumptions about firms' behavior: a quantity setting or Cournot approach and a price setting or Bertrand approach. It is well known that, when goods are substitutes, the Bertrand equilibrium is more efficient than the Cournot equilibrium (see Singh and Vives,

*Corresponding author.
E-mail address: yasuhito@tamacc.chuo-u.ac.jp (Y. Tanaka).

0167-7187/01/$ – see front matter
PII: S0167-7187(00)00058-8

1984; Cheng, 1985; and Vives, 1985). These analyses are based on the Nash equilibrium concept. In this paper we present an evolutionary game theoretic analysis of oligopoly. We consider an oligopoly with differentiated goods composed of two groups (or populations) of firms, one group of low cost firms and one of high cost firms. We study finite population evolutionarily stable strategies (ESSs) defined by Schaffer (1988) and stochastically stable states (or long run equilibria in terms of Kandori et al. (1993)) for a model of evolution with an imitative rule of strategy choice with mutations. A stochastically stable state is a state where most of the time is spent in the long run when the probability of mutation becomes very small. Our formulation of a model of evolution with an imitative rule of strategy choice follows Robson and Vega-Redondo (1996) and Vega-Redondo (1997).

Vega-Redondo (1997) studied imitative behavior in a symmetric oligopoly with a homogeneous good, and showed that Walrasian behavior (profit maximization given the market clearing price) is a stochastically stable strategy. Tanaka (1999a) extended the result of Vega-Redondo (1997) to a case of *asymmetric* homogeneous oligopoly with low cost and high cost firms, and showed that under the assumption that marginal cost is increasing a stochastically stable outcome is the competitive (Walrasian) output for each group of firms. On the other hand, Tanaka (1998) showed that stochastically stable states coincide for a quantity setting oligopoly and a price setting oligopoly in a *symmetric* oligopoly with differentiated goods under linear demand functions. Rhode and Stegeman (2000) analyzed Darwinian dynamics of a symmetric, differentiated duopoly with linear demand functions. They showed that firms' strategy choices cluster around a strategy profile that is not a one-shot Nash equilibrium, and this profile is invariant under a class of transformations of the strategy spaces (Bertrand vs. Cournot). They considered a stationary distribution of a Markov chain with large and frequent mutations. By contrast, we consider a limit of a stationary distribution of a Markov chain as mutations vanish according to the formulation by Robson and Vega-Redondo (1996) and Vega-Redondo (1997).

Schaffer (1988) proposed a concept of an evolutionarily stable strategy (ESS) for a finite (or small) population as a generalization of the standard ESS concept for an infinite (or large) population by Maynard Smith (1982). We call it a *finite population ESS*. He showed that a finite population ESS is not generally a Nash equilibrium strategy. In Schaffer (1989) he applied this concept to an economic game, and showed that the strategy which survives in economic natural selection is a relative, not absolute, payoff maximizing strategy. He considered the following survival rule. Firms are born with strategies and cannot change their strategies in response to changing circumstances. At the end of each period, if the payoff of firm i is larger than the payoff of firm j, the probability that firm i survives in the next period is larger than the probability that firm j survives in the next period. Alternatively we consider that the survival rule operates on strategies, not firms, and the proportion of successful strategies in the population grows by firms' imitation of strategies.[1]

In this paper we consider the following model of an asymmetric oligopoly with differentiated goods. The low cost firms and the high cost firms exist in two distinct regions, and the difference of their costs reflect the difference of conditions for production between two regions such as the distance from the main market and infrastructure.[2] Every firm can observe decisions of other firms in its group, but does not know the exact form of demand functions, and cannot compute its best response to other firms' strategies because of ignorance of demand functions or absence of ability to compute a best response. On the other hand, every firm knows that the cost functions of all firms in its group are the same. When all firms in each group choose the same strategy, denoting it by s_1, their profits are equal, and they do not have incentive to change their strategies. Now, suppose that one firm (the mutant firm) experiments a different strategy, s_2. If this firm makes higher profit than the rest of the firms, they will wish to imitate the mutant firm's success. On the other hand, if the mutant firm makes lower profit than the rest of the firms, they will not wish to imitate the mutant firm's failure, and in fact the mutant firm will wish to switch s_2 to s_1. If, starting from s_1, experimenting always leads to lower profit for the mutant firm than for the rest of the firms, then s_1 is an ESS.

The mechanism of an imitative strategy choice will be explained in Section 3. Some recent papers such as Robson and Vega-Redondo (1996) and Vega-Redondo (1997) considered a model of evolution with an imitative strategy choice. On the other hand, some other papers such as Kandori and Rob (1995) and (1998), and Galesloot and Goyal (1997) considered a model of evolution based on best response dynamics. In best response dynamics each player chooses a strategy in period $t+1$ which is a best response to other players' strategies in period t. Thus players must know the whole payoff structure of the game, and be able to compute their best responses. While in imitation dynamics, players simply mimic successful strategies of other players. We think that imitation dynamics is more appropriate

[1] Hansen and Samuelson (1988) also presented analyses about evolution in economic games. They showed that with a small number of firms a surviving strategy in economic natural selection, they called such a strategy a *universal survival strategy*, is not a Nash equilibrium strategy. Their universal survival strategy is essentially equivalent to Schaffer's finite population evolutionarily stable strategy. They said, 'In real-world competition, firms will be uncertain about the profit outcomes of alternative strategies. This presents an obvious obstacle to instantaneous optimization. Instead, firms must search for and learn about more profitable strategies. As Alchian (1950) emphasizes, an important mechanism for such a search depends on a comparison of observed profitability across the strategies used by market participants. That is, search for better strategies is based on *relative* profit comparisons'. For a more recent analysis of imitation behavior, see Schlag (1998).

[2] This interpretation was suggested by a referee.

than best response dynamics for an economic game with boundedly rational players.

We are concerned with showing that, under linear demand functions, in a quantity setting oligopoly the finite population ESS output of the low cost firms and that of the high cost firms are stochastically stable strategies, and in a price setting oligopoly the finite population ESS price of the low cost firms and that of the high cost firms are stochastically stable strategies. Further, we will show that the stochastically stable state, in which all firms in each group choose the stochastically stable strategy, in a quantity setting oligopoly and that in a price setting oligopoly coincide.

In the next section we consider finite population evolutionarily stable strategies. In Section 3 we consider stochastically stable states, and show the main results. The last section contains concluding remarks.

2. Finite population evolutionarily stable strategies

2.1. A quantity setting oligopoly

There is an oligopoly in which N firms produce differentiated goods. There are two groups of firms, one group of low cost firms and one of high cost firms. The number of the low cost firms is n_1, and the number of the high cost firms is n_2. n_1 and n_2 are integers which are not smaller than 2. We assume no entry and exit, and that n_1 and n_2 are fixed. Denote $N = n_1 + n_2$. Let x_i and p_i be the output and price of the ith low cost firm's good, and let y_i and q_i be the output and price of the ith high cost firm's good. Then, the inverse demand functions are

$$p_i = a - x_i - b \sum_{j=1, j\neq i}^{n_1} x_j - b \sum_{j=1}^{n_2} y_j,\ i = 1, 2, \ldots, n_1, \tag{1}$$

and

$$q_i = a - y_i - b \sum_{j=1, j\neq i}^{n_2} y_j - b \sum_{j=1}^{n_1} x_j,\ i = 1, 2, \ldots, n_2, \tag{2}$$

where $a > 0$ and $0 < b < 1$.

The marginal (and average) cost of the low cost firms is c_l (> 0), and that of the high cost firms is c_h ($> c_l$). The firms in two groups exist in two distinct regions, and the difference of the costs reflect the difference of conditions for production between two regions such as distance from the main market and infrastructure. There is no fixed cost.

The profits of the low cost firms are

$$\pi_i^l(\mathbf{x},\mathbf{y}) = \left(a - x_i - b\sum_{j=1, j\neq i}^{n_1} x_j - b\sum_{j=1}^{n_2} y_j\right)x_i - c_l x_i,\ i = 1, 2, \ldots, n_1, \quad (3)$$

and the profits of the high cost firms are

$$\pi_i^h(\mathbf{x},\mathbf{y}) = \left(a - y_i - b\sum_{j=1, j\neq i}^{n_2} y_j - b\sum_{j=1}^{n_1} x_j\right)y_i - c_h y_i,\ i = 1, 2, \ldots, n_2,$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_{n_1}) \text{ and } \mathbf{y} = (y_1, y_2, \ldots, y_{n_2}).$$

We consider an evolutionary game in which N firms repeatedly play an oligopoly stage game. In this evolutionary game the population is N, and the stage game is also an N players game. Thus it is a so-called *playing the fields model*. Strategies for the firms are their outputs. The firms repeatedly play the stage game in each period, and may change their strategies between one period and the next period. This dynamic problem is treated in the next section. In this section we consider finite population evolutionarily stable strategies in the stage game.

Consider a state in which all of the low cost firms choose x^*. If, when one of the low cost firms (a mutant firm) chooses a different strategy x', the profits of the firms who choose x^* are larger than the profit of the mutant firm given the outputs of the high cost firms, then x^* is the finite population evolutionarily stable strategy (ESS) *relative* to the outputs of the high cost firms.[3] Formally, x^* is the finite population ESS if

$$\pi_i^l(\mathbf{x},\mathbf{y}) > \pi_j^l(\mathbf{x},\mathbf{y}) \text{ for each } i \neq j \quad (4)$$

where $x_j \neq x^*$, and $x_i = x^*$ for all $i \neq j$ in $\mathbf{x}$. Without loss of generality the mutant firm is represented as firm j. According to Schaffer (1988) we can find the finite population ESS as the solution of the following problem,[4]

$$x^* = \underset{x'}{\operatorname{argmax}}\,\varphi, \quad (5)$$

where

$$\varphi = \pi_j^l(\mathbf{x},\mathbf{y}) - \pi_i^l(\mathbf{x},\mathbf{y}) \text{ with } x_j = x', \text{ and } x_i = x^* \text{ for all } i \neq j \text{ in } \mathbf{x}.$$

If there is a unique maximizer x' in (5) given x^*, then x^* satisfies (4) since φ has

[3] Schaffer's definition is weaker. He defines that x^* is the finite population ESS if (4) is satisfied with weak inequality. We adopt the definition with strong inequality. About the definition of a finite population ESS, see Crawford (1991).

[4] Eq. (5) means that a finite population ESS is a strategy which maximizes the firms' relative profit. Therefore, as stated in Schaffer (1988) and (1989), a finite population ESS is equivalent to a so-called *beat the average* strategy in Shubik and Levitan (1980).

the maximum value, which is zero, only when $x' = x^*$. Substituting (3) into φ, replacing x_i and x_j, respectively, by x^* and x', yields

$$\varphi = \left[a - x' - (n_1 - 1)bx^* - b \sum_{i=1}^{n_2} y_i \right] x' - c_l x' - \left[a - x^* - bx' - (n_1 - 2)bx^* - b \sum_{i=1}^{n_2} y_i \right] x^* + c_l x^*.$$

The condition for maximization of φ with respect to x' is

$$a - 2x' - (n_1 - 1)bx^* - b\sum_{i=1}^{n_2} y_i - c_l + bx^* = 0. \tag{6}$$

The value of x^* depends on $\Sigma_{i=1}^{n_2} y_i$. Denote x^* which is derived from (5) when all of the high cost firms choose y by $x^*(y)$. We have

$$x^*(y) = \frac{a - c_l - n_2 by}{2 + (n_1 - 2)b}. \tag{7}$$

Call $x^*(y)$ *the ESS output for the low cost firms relative to* y.

The finite population ESS for the high cost firms is similarly defined, and we obtain *the ESS output for the high cost firms relative to* x as follows,

$$y^*(x) = \frac{a - c_h - n_1 bx}{2 + (n_2 - 2)b}. \tag{8}$$

Simultaneously solving (7) and (8), the true finite population ESS outputs for the low cost firms and the high cost firms are obtained as follows,

$$x^{**} = \frac{a - c_l}{2 + (N - 2)b} - \frac{n_2 b(c_l - c_h)}{2(1 - b)[2 + (N - 2)b]}, \tag{9}$$

and

$$y^{**} = \frac{a - c_h}{2 + (N - 2)b} - \frac{n_1 b(c_h - c_l)}{2(1 - b)[2 + (N - 2)b]}. \tag{10}$$

Since $c_l < c_h$, we easily find $x^{**} > y^{**}$.

Substituting (9) and (10) into (1), the price of the low cost firms' goods with the ESS outputs for the firms in both groups is obtained as follows,

$$p_i = a - [1 + (n_1 - 1)b]x^{**} - n_2 by^{**} = \frac{(1 - b)a + [1 + (N - 1)b]c_l}{2 + (N - 2)b} - \frac{n_2 b(c_l - c_h)}{2[2 + (N - 2)b]}. \tag{11}$$

Similarly, substituting (9) and (10) into (2), the price of the high cost firms' goods with the ESS outputs is obtained as follows,

$$q_i = a - [1 + (n_2 - 1)b]y^{**} - n_1 bx^{**}$$
$$= \frac{(1-b)a + [1 + (N-1)b]c_h}{2 + (N-2)b} - \frac{n_1 b(c_h - c_l)}{2[2 + (N-2)b].} \tag{12}$$

2.2. A price setting oligopoly

Next consider a price setting oligopoly. The inverse demand functions for the goods are the same as those in the quantity setting case, which are represented in (1) and (2). From the inverse demand functions we obtain the direct demand functions for the ith low cost firm and the ith high cost firm as follows (see Appendix A),

$$x_i = \frac{1}{(1-b)[1 + (N-1)b]} \left\{ (1-b)a - [1 + (N-2)b]p_i + b \sum_{j=1, j\neq i}^{n_1} p_j + b \sum_{j=1}^{n_2} q_j \right\}, \; i = 1, 2, \ldots, n_1, \tag{13}$$

and

$$y_i = \frac{1}{(1-b)[1 + (N-1)b]} \left\{ (1-b)a - [1 + (N-2)b]q_i + b \sum_{j=1, j\neq i}^{n_2} q_j + b \sum_{j=1}^{n_1} p_j \right\}, \; i = 1, 2, \ldots, n_2. \tag{14}$$

By similar procedures to the case of a quantity setting oligopoly, we obtain *the* (*finite population*) *ESS price for the low cost firms relative to* q as follows,

$$p^*(q) = \frac{(1-b)a + [1 + (N-1)b]c_l + n_2 bq}{2 + (N + n_2 - 2)b}. \tag{15}$$

And we obtain *the ESS price for the high cost firms relative to* p as follows,

$$q^*(p) = \frac{(1-b)a + [1 + (N-1)b]c_h + n_1 bp}{2 + (N + n_1 - 2)b}. \tag{16}$$

Simultaneously solving (15) and (16), the true finite population ESS prices for the low cost firms and the high cost firms are obtained as follows,

$$p^{**} = \frac{2(1-b)a + [2 + (N-2)b + n_1 b]c_l + n_2 bc_h}{2[2 + (N-2)b]}$$
$$= \frac{(1-b)a + [1 + (N-1)b]c_l}{2 + (N-2)b} - \frac{n_2 b(c_l - c_h)}{2[2 + (N-2)b]}, \tag{17}$$

and

$$q^{**} = \frac{2(1-b)a + [2+(N-2)b + n_2 b]c_h + n_1 b c_l}{2[2+(N-2)b]}$$
$$= \frac{(1-b)a + [1+(N-1)b]c_h}{2+(N-2)b} - \frac{n_1 b(c_h - c_l)}{2[2+(N-2)b]}. \tag{18}$$

Since

$$p^{**} - q^{**} = \frac{[1+(N-1)b](c_l - c_h)}{2+(N-2)b} - \frac{Nb(c_l - c_h)}{2[2+(N-2)b]}$$
$$= \frac{[2+(N-2)b](c_l - c_h)}{2[2+(N-2)b]} < 0,$$

we have $p^{**} < q^{**}$.

Substituting (17) and (18) into (13), the output of the low cost firms with the ESS prices for the firms in both groups is obtained as follows,

$$x_i = \frac{1}{(1-b)[1+(N-1)b]}\{(1-b)a - [1+(N-2)b]p^{**} + (n_1 - 1)bp^{**} + n_2 b q^{**}\}$$
$$= \frac{a - c_l}{2+(N-2)b} - \frac{n_2 b(c_l - c_h)}{2(1-b)[2+(N-2)b]}. \tag{19}$$

Similarly, substituting (17) and (18) into (14), the output of the high cost firms with the ESS prices is obtained as follows,

$$y_i = \frac{1}{(1-b)[1+(N-1)b]}\{(1-b)a - [1+(N-2)b]q^{**} + (n_2 - 1)bq^{**} + n_1 b p^{**}\}$$
$$= \frac{a - c_h}{2+(N-2)b} - \frac{n_1 b(c_h - c_l)}{2(1-b)[2+(N-2)b]}. \tag{20}$$

2.3. The equivalence of the finite population ESS in a quantity setting oligopoly and that in a price setting oligopoly

From (9), (10), (19) and (20) we find that in each group of the firms the finite population ESS output in a quantity setting oligopoly is equal to the output with the ESS prices in a price setting oligopoly. Similarly, from (11), (12), (17) and (18) we find that in each group of the firms the finite population ESS price in a price setting oligopoly is equal to the price with the ESS outputs in a quantity setting oligopoly.

Therefore we obtain the following result.

Theorem 1. *The finite population ESS in a quantity setting oligopoly and that in a price setting oligopoly are equivalent for each group of the firms.*

3. Stochastically stable states

3.1. A quantity setting oligopoly

In this section we will show that the finite population ESS output for each group of the firms obtained in the previous section is a stochastically stable strategy for a model of evolution with an imitative rule of strategy choice with mutations. Kandori et al. (1993), Kandori and Rob (1995), Robson and Vega-Redondo (1996) and Vega-Redondo (1997) presented analyses of stochastically stable states in evolutionary games. In our model, N players (firms) play an oligopoly game in each period. According to Robson and Vega-Redondo (1996) and Vega-Redondo (1997) we consider an imitative strategy choice by firms as follows. In period $t+1$ every low cost (respectively high cost) firm has a chance with positive probability less than one to change its strategy to the strategy which achieved the highest profit among strategies chosen by the low cost (respectively high cost) firms in period t. If the strategy of one firm in period t achieved the strictly highest profit in its group, this firm does not change its strategy. If in period t the highest profit was attained by two or more firms in one of the groups even when they chose different strategies, then in period $t+1$ every firm in this group may choose either strategy among the strategies which attained the highest profit in period t. If all of the firms in one of the groups chose the same strategy in period t, since in such a state this strategy achieved the strictly highest profit, all of the firms in this group do not change their strategies.

As in Vega-Redondo (1997) we assume that the firms in each group must choose their outputs from a finite grid $\Gamma = \{0, \delta, 2\delta, \dots, v\delta\}$ where $\delta > 0$ and $v \in \mathbb{N}$ are arbitrary. We call δ *the grid step*. It is required that the finite population ESS outputs x^{**} and y^{**} belong to this grid. δ can be arbitrarily small to make the grid sufficiently fine. States of the imitation dynamics are identified with firms' output profiles. The state space is denoted by Ω which is equal to Γ^N. Denote the transition matrix of this dynamics by $T(\omega, \omega')$, and by $T^{(m)}(\omega, \omega')$ the corresponding m-step transition matrix, where $\omega, \omega' \in \Omega$.

In addition to this dynamic adjustment, there is a random mutation. In each period, every firm switches (mutates) its strategy with probability ε. Mutation may be interpreted as experimentation of a new strategy by some firms. All strategies may be chosen with positive probability. Thus the complete dynamic is an ergodic Markov chain, and it has a unique stationary distribution. Consider the limit of the stationary distribution of the Markov chain as $\varepsilon \to 0$. Stochastically stable states are states which are assigned positive probability in the limit.[5]

We define a *limit set* of the dynamics without mutation. A set A is a limit set if this set is closed under the finite chains of positive probability transitions. That is,

1. $\forall \omega \in A$, $\forall \omega' \notin A$, $T(\omega, \omega') = 0$.
2. $\forall \omega \in A$, $\omega' \in A$, $\exists m \in \mathbb{N}$ such that $T^{(m)}(\omega, \omega') > 0$.

If in period t some firms choose different strategies in one of the groups, at least one firm has a chance to change its strategy with positive probability without mutation. Therefore, such a state cannot be included in any limit set, and in any state included in some limit set all of the firms in each group must choose the same strategy. On the other hand, in any state in which all of the firms in each group choose the same strategy, no firm has incentive to change its strategy except for mutation. Accordingly, a limit set is identified as a set which includes a single state in which all of the firms in each group choose the same strategy. We need no mutation to move from any state, which is not included in a limit set, to a state in some limit set. Thus a stochastically stable state must be in some limit set. According to Kandori and Rob (1995) we consider a reduced Markov chain defined on the limit sets. Denote the state in which all low cost firms choose x and all high cost firms choose y by $\omega(x, y)$. The number of the states (including the states where $x = 0$ or $y = 0$) is $(v + 1)^2$.

Now construct a directed graph which contains the directed paths from $\omega(x, y)$ to $\omega(x^{**}, y^{**})$ for all $(x, y) \neq (x^{**}, y^{**})$, and the path from $\omega(x^{**}, y^{**})$ to some state $\omega(x', y')$, $(x', y') \neq (x^{**}, y^{**})$. Denote this graph by G.

Eliminating the path from $\omega(x, y)$ to $\omega(x^{**}, y^{**})$ (or the path from $\omega(x^{**}, y^{**})$ to $\omega(x', y')$) in G we get an $\omega(x, y)$-tree (or $\omega(x^{**}, y^{**})$-tree). An $\omega(x, y)$-tree is a collection of directed branches $(\omega(x_1, y_1), \omega(x_2, y_2))$, $\omega(x_2, y_2)$ is the successor of $\omega(x_1, y_1)$, where, (1) except for $\omega(x, y)$, each state has a unique successor, and, (2) there is no closed loop.[6] Denote the total number of mutations in an $\omega(x, y)$-tree by $C(x, y)$. Based on the results in Freidlin and Wentzel (1984), in their Proposition 4 Kandori and Rob (1995) showed that the stochastically stable states comprise the states having minimum $C(x, y)$.

From the arguments in the previous section we can see that, since x^{**} and y^{**} are the finite population ESS outputs, one mutation is not sufficient and we need at

[5] This adjustment process is the same as that in Robson and Vega-Redondo (1996) and Vega-Redondo (1997). It has a stochastic nature even without mutation since every firm has a chance to change its strategy independently with some positive probability, and the number of the firms in each group who change their strategies in period $t + 1$ to the best strategy in period t is a stochastic variable without mutation. In period $t + 1$ all of the firms may choose the best strategy in period t with strictly positive probability.

[6] For more details about a tree see Kandori et al. (1993), Vega-Redondo (1996, 1997) and Young (1998).

least two mutations to move from the state $\omega(x^{**}, y^{**})$ to any other state. Therefore $C(x, y) \geqq (v+1)^2$ for $\omega(x, y) \neq \omega(x^{**}, y^{**})$.

Next we have to examine how many mutations we need to move from a state $\omega(x, y)$, $(x, y) \neq (x^{**}, y^{**})$, to the state $\omega(x^{**}, y^{**})$. Consider the state in which all low cost firms choose x' and all high cost firms choose y', where $x' \neq x^*(y')$. $x^*(y')$ is the ESS output relative to y' defined in the previous section. If one of the low cost firms (a mutant firm) chooses $x^*(y')$, the profit of this firm is

$$\pi_i^l(\mathbf{x}, \mathbf{y}) = [a - x^*(y') - (n_1 - 1)bx' - n_2 by']x^*(y') - c_l x^*(y'). \tag{21}$$

And the profits of the other low cost firms are

$$\pi_j^l(\mathbf{x}, \mathbf{y}) = [a - x' - bx^*(y') - (n_1 - 2)bx' - n_2 by']x' - c_l x' \text{ for } j \neq i, \tag{22}$$

where $x_i = x^*(y')$, and $x_j = x' \neq x^*(y')$ for all $j \neq i$ in $\mathbf{x}$. Comparing (21) and (22),

$$\pi_i^l(\mathbf{x}, \mathbf{y}) - \pi_j^l(\mathbf{x}, \mathbf{y}) = [a - c_l - n_2 by' - x^*(y') - x' - (n_1 - 2)bx'](x^*(y') - x')). \tag{23}$$

From (7) we have $a - c_l - n_2 by' = [2 + (n_1 - 2)b]x^*(y')$. Substituting this into (23) yields

$$\pi_i^l(\mathbf{x}, \mathbf{y}) - \pi_j^l(\mathbf{x}, \mathbf{y}) = [1 + (n_1 - 2)b](x^*(y') - x')^2. \tag{24}$$

So long as $x^*(y') \neq x'$, this is strictly positive. Thus the profit of the mutant firm is larger than the profits of the other firms. It implies that we can move from any state $\omega(x', y')$, $x' \neq x^*(y')$, to the state $\omega(x^*(y'), y')$ by only one mutation.

Similarly, at the state $\omega(x', y')$ where $y' \neq y^*(x')$, if one of the high cost firms chooses $y^*(x')$, its profit is

$$\pi_i^h(\mathbf{x}, \mathbf{y}) = [a - y^*(x') - (n_2 - 1)by' - n_1 bx']y^*(x') - c_h y^*(x').$$

And the profits of the other high cost firms are

$$\pi_j^h(\mathbf{x}, \mathbf{y}) = [a - y' - by^*(x') - (n_2 - 2)by' - n_1 bx']y' - c_h y' \text{ for } j \neq i,$$

where $y_i = y^*(x')$, and $y_j = y' \neq y^*(x')$ for all $j \neq i$ in $\mathbf{y}$. Similarly to (24) we obtain

$$\pi_i^h(\mathbf{x}, \mathbf{y}) - \pi_j^h(\mathbf{x}, \mathbf{y}) = [1 + (n_2 - 2)b](y^*(x') - y')^2.$$

So long as $y^*(x') \neq y'$, this is strictly positive. It implies that we can move from any state $\omega(x', y')$, $y' \neq y^*(x')$, to the state $\omega(x', y^*(x'))$ by only one mutation.

Subtracting (9) from (7) side by side, we obtain

$$x^*(y) - x^{**} = -\frac{n_2 b}{2 + (n_1 - 2)b}(y - y^{**}). \tag{25}$$

Similarly, subtracting (10) from (8) side by side,

$$y^*(x) - y^{**} = -\frac{n_1 b}{2 + (n_2 - 2)b}(x - x^{**}). \tag{26}$$

From (25) and (26) we get

$$x^*(y) - x^{**} = \frac{n_1 n_2 b^2}{[2 + (n_1 - 2)b][2 + (n_2 - 2)b]}(x - x^{**}),$$

and

$$y^*(x) - y^{**} = \frac{n_1 n_2 b^2}{[2 + (n_1 - 2)b][2 + (n_2 - 2)b]}(y - y^{**}).$$

Since $1 - b > 0$, we have $2 + (n_1 - 2)b > n_1 b$, and $2 + (n_2 - 2)b > n_2 b$. Then we find

$$0 < \frac{n_1 n_2 b^2}{[2 + (n_1 - 2)b][2 + (n_2 - 2)b]} < 1.$$

Thus, there is a sequence of the states which starts from $\omega(x', y')$, passes through $\omega(x^*(y'), y')$, through $\omega(x^*(y'), y^*(x^*(y')))$, through $\omega(x^*(y^*(x^*(y'))), y^*(x^*(y')))$ and so on. This sequence approaches to $\omega(x^{**}, y^{**})$. If the grid of the firms' outputs is sufficiently fine, this sequence of the states converges to some limit state which is in some neighborhood of $\omega(x^{**}, y^{**})$. Then, we obtain the following lemma.

Lemma 1. *For every $\eta > 0$, there is some $\bar{\delta} > 0$ such that if the grid step δ is not larger than $\bar{\delta}$, the stochastically stable state of the corresponding finite-state process is in some η-neighborhood of the ESS state $\omega(x^{**}, y^{**})$.*

Denote a limit state of some sequence by $\omega(x^0, y^0)$, which may be different for different starting states. As $\delta \to 0$, any $\omega(x^0, y^0)$ converges to $\omega(x^{**}, y^{**})$. Then we find $C(x^{**}, y^{**}) = (v + 1)^2 - 1$, which is the number of the states other than $\omega(x^{**}, y^{**})$ given δ. It is smaller than $C(x, y)$ for $(x, y) \neq (x^{**}, y^{**})$. Thus, we obtain the following result.

Corollary 1. $\omega(x^{**}, y^{**})$ *is a stochastically stable state.*

3.2. A price setting oligopoly

A framework for the analysis of stochastically stable states in a price setting oligopoly is parallel to the analysis in a quantity setting oligopoly. We assume that the firms in both groups must choose their prices from a finite grid $\Gamma' = \{0, \delta', 2\delta', \ldots, v'\delta'\}$ where $\delta' > 0$ and $v' \in \mathbb{N}$ are arbitrary. It is required that the finite population ESS prices p^{**} and q^{**} belong to this grid. The adjustment process and limit sets are similarly defined. A stochastically stable state must be in some limit set. We consider a reduced Markov chain defined on the limit sets. Denote the state in which all low cost firms choose p and all high cost firms choose q by $\omega(p, q)$. The number of the states (including the states where $p=0$ or $q=0$) is $(v'+1)^2$. $C(p, q)$ is similarly defined, and we obtain $C(p, q) \geqq (v'+1)^2$ for $\omega(p, q) \neq \omega(p^{**}, q^{**})$.

Similarly to the case of a quantity setting oligopoly we can show

Lemma 2. *For every $\eta' > 0$, there is some $\bar{\delta}' > 0$ such that if the grid step δ' is not larger than $\bar{\delta}'$, the stochastically stable state of the corresponding finite-state process is in some η'-neighborhood of the ESS state $\omega(p^{**}, q^{**})$*

and

Corollary 2. *$\omega(p^{**}, q^{**})$ is a stochastically stable state.*

Proof. See Appendix B.

As we have shown in Theorem 1, the finite population ESS in a quantity setting oligopoly and that in a price setting oligopoly are equivalent. Therefore we obtain the following conclusion.

Theorem 2. *The stochastically stable state in a quantity setting oligopoly and that in a price setting oligopoly coincide.*

4. Concluding remarks

Let us consider the difference between stochastically stable strategies and Nash equilibrium strategies. In a quantity setting oligopoly, when the goods are substitutes, each firm determines its output with a conjecture that if it increases its output, the prices of the other firms' goods will fall. Such responses reduce the price of its good. On the other hand, in a price setting oligopoly, each firm determines the price of its good with a conjecture that if it reduces the price of its good, the outputs of the other firms will decrease. Such responses increase the demand for its good. Then the firms in a price setting oligopoly should be more aggressive than those in a quantity setting oligopoly. These are why the Nash equilibrium in a quantity setting oligopoly and that in a price setting oligopoly are different.

With imitation dynamics what matters is that a quantity increase in a quantity setting oligopoly or a price increase in a price setting oligopoly, as long as it raises a firm's profit *relatively* to those of the other firms, it will be imitated. This suggests why both games lead to the same outcome.

In this paper we have considered only a case of linear demand functions. In another paper, Tanaka (1999b), we have analyzed a stochastically stable state in a symmetric oligopoly with differentiated goods under general demand functions, and have shown that a strategy which, with one mutation, can invade any state where all firms choose the same strategy other than this strategy is a stochastically stable strategy in both quantity setting and price setting oligopolies. It is different from but closely related to a finite population ESS. In this paper we have sacrificed generality for extending the arguments to an asymmetric oligopoly.

Acknowledgements

I wish to thank anonymous referees for very helpful comments.

Appendix A. Derivations of (13) and (14)

Summing up both sides of (1) over $i = 1, 2, \ldots, n_1$ yields

$$\sum_{i=1}^{n_1} p_i = n_1 a - \sum_{i=1}^{n_1} x_i - (n_1 - 1)b \sum_{i=1}^{n_1} x_i - n_1 b \sum_{i=1}^{n_2} y_i$$

$$= n_1 a - [1 + (n_1 - 1)b] \sum_{i=1}^{n_1} x_i - n_1 b \sum_{i=1}^{n_2} y_i.$$

Similarly from (2) we have

$$\sum_{i=1}^{n_2} q_i = n_2 a - [1 + (n_2 - 1)b] \sum_{i=1}^{n_2} y_i - n_2 b \sum_{i=1}^{n_1} x_i.$$

Adding these two equations side by side yields

$$\sum_{i=1}^{n_1} p_i + \sum_{i=1}^{n_2} q_i = (n_1 + n_2)a - [1 + (n_1 + n_2 - 1)b]\left(\sum_{i=1}^{n_1} x_i + \sum_{i=1}^{n_2} y_i\right).$$

From this

$$\sum_{i=1}^{n_1} x_i + \sum_{i=1}^{n_2} y_i = \frac{1}{1 + (n_1 + n_2 - 1)b}\left[(n_1 + n_2)a - \sum_{i=1}^{n_1} p_i - \sum_{i=1}^{n_2} q_i\right]. \tag{A.1}$$

The inverse demand function for the ith low cost firm is rewritten as

$$p_i = a - (1 - b)x_i - b\sum_{j=1}^{n_1} x_j - b\sum_{j=1}^{n_2} y_j.$$

From this

$$x_i = \frac{1}{1-b}\left[a - p_i - b\left(\sum_{j=1}^{n_1} x_j + \sum_{j=1}^{n_2} y_j\right)\right].$$

Therefore, from (A.1) the direct demand function for the ith low cost firm is derived as follows,

$$\begin{aligned} x_i &= \frac{1}{(1-b)[1+(n_1+n_2-1)b]}\Bigg\{(1-b)a - [1+(n_1+n_2-1)b]p_i \\ &\quad + b\sum_{j=1}^{n_1} p_j + b\sum_{j=1}^{n_2} q_j\Bigg\} \\ &= \frac{1}{(1-b)[1+(N-1)b]}\Bigg\{(1-b)a - [1+(N-2)b]p_i + b\sum_{j=1, j\neq i}^{n_1} p_j \\ &\quad + b\sum_{j=1}^{n_2} q_j\Bigg\}, \quad i = 1, 2 \ldots, n_1. \end{aligned}$$

Similarly, the demand function for the ith high cost firm is derived as follows,

$$\begin{aligned} y_i &= \frac{1}{(1-b)[1+(N-1)b]}\Bigg\{(1-b)a - [1+(N-2)b]q_i + b\sum_{j=1, j\neq i}^{n_2} q_j \\ &\quad + b\sum_{j=1}^{n_1} p_j\Bigg\}, i = 1, 2, \ldots, n_2. \end{aligned}$$

Appendix B. Proofs of Lemma 2 and Corollary 2

Let us construct a directed graph which contains the directed paths from $\omega(p, q)$ to $\omega(p^{**}, q^{**})$ for all $(p, q) \neq (p^{**}, q^{**})$, and the path from $\omega(p^{**}, q^{**})$ to some state $\omega(p', q')$, $(p', q') \neq (p^{**}, q^{**})$. Denote this graph by G'.

Eliminating the path from $\omega(p, q)$ to $\omega(p^{**}, q^{**})$ to $\omega(p', q')$ (or the path from $\omega(p^{**}, q^{**})$ to $\omega(p', q')$) in G' we get an $\omega(p, q)$-tree (or $\omega(p^{**}, q^{**})$-tree). Denote the total number of mutations in an $\omega(p, q)$-tree by $C(p, q)$. The stochastically stable states comprises the states having a minimum $C(p, q)$.

From the arguments in Section 2 we can see that, since p^{**} and q^{**} are the finite population ESS prices, one mutation is not sufficient and we need at least two mutations to move from the state $\omega(p^{**}, q^{**})$ to any other state. Therefore $C(p, q) \geqq (v'+1)^2$ for $\omega(p, q) \neq \omega(p^{**}, q^{**})$.

Next we have to examine how many mutations we need to move from a state $\omega(p, q)$, $(p, q) \neq (p^{**}, q^{**})$, to the state $\omega(p^{**}, q^{**})$. Consider the state in which all low cost firms choose p' and all high cost firms choose q', where

$p' \neq p^*(q')$, which is the ESS price for the low cost firms relative to q'. If one of the low cost firms (a mutant firm) chooses $p^*(q')$, the profit of this firm is

$$\pi_i^l(\mathbf{p}, \mathbf{q}) = \frac{1}{(1-b)[1+(N-1)b]}\{(1-b)a - [1+(N-2)b]p^*(q') + (n_1-1)bp' + n_2bq'\}(p^*(q') - c_l). \tag{B.1}$$

And the profits of the other low cost firms are

$$\pi_j^l(\mathbf{p}, \mathbf{q}) = \frac{1}{(1-b)[1+(N-1)b]}\{(1-b)a - [1+(N-2)b]p' + bp^*(q') + (n_1-2)bp' + n_2bq'\}(p' - c_l) \text{ for } j \neq i, \tag{B.2}$$

where $p_i = p^*(q')$, and $p_j = p' \neq p^*(q')$ for all $j \neq i$ in $\mathbf{p}$. Comparing (B.1) and (B.2),

$$\pi_i^l(\mathbf{p}, \mathbf{q}) - \pi_j^l(\mathbf{p}, \mathbf{q}) = \frac{1}{(1-b)[1+(N-1)b]}\{(1-b)a - [1+(N-2)b](p^*(q') + p') + (n_1-2)bp' + n_2bq' + [1+(N-1)b]c_l\}(p^*(q') - p'). \tag{B.3}$$

From (15) we have $(1-b)a + [1+(N-1)b]c_l + n_2bq' = [2+(N+n_2-2)b]p^*(q')$. Substituting this into (B.3) yields

$$\pi_i^l(\mathbf{p}, \mathbf{q}) - \pi_j^l(\mathbf{p}, \mathbf{q}) = \frac{1}{(1-b)[1+(N-1)b]}(n_2+1)(p^*(q') - p')^2. \tag{B.4}$$

So long as $p^*(q') \neq p'$, this is strictly positive. Thus the profit of the mutant firm is larger than the profits of the other firms. It implies that we can move from any state $\omega(p', q')$, $p' \neq p^*(q')$, to the state $\omega(p^*(q'), q')$ by only one mutation.

Similarly, at the state $\omega(p', q')$ where $q' \neq q^*(p')$, if one of the high cost firms chooses $q^*(p')$, its profit is

$$\pi_i^h(\mathbf{p}, \mathbf{q}) = \frac{1}{(1-b)[1+(N-1)b]}\{(1-b)a - [1+(N-2)b]q^*(p') + (n_2-1)bq' + n_1bp'\}(q^*(p') - c_h).$$

And the profits of the other high cost firms are

$$\pi_j^h(\mathbf{p}, \mathbf{q}) = \frac{1}{(1-b)[1+(N-1)b]}\{(1-b)a - [1+(N-2)b]q' + bq^*(p') + (n_2-2)bq' + n_1bp'\}(q' - c_h) \text{ for } j \neq i,$$

where

$$q_i = q^*(p'), \text{ and } q_j = q' \neq q^*(p') \text{ for all } j \neq i \text{ in } \mathbf{q}.$$

Similarly to (B.4) we obtain

$$\pi_i^h(\mathbf{p},\mathbf{q}) - \pi_j^h(\mathbf{p},\mathbf{q}) = \frac{1}{(1-b)[1+(N-1)b]}(n_1+1)(q^*(p') - q')^2.$$

So long as $q^*(p') \neq q'$, this is strictly positive. It implies that we can move from any state $\omega(p', q')$, $q' \neq q^*(p')$, to the state $\omega(p', q^*(p'))$ by only one mutation.

Subtracting (17) from (15) side by side, we obtain

$$p^*(q) - p^{**} = -\frac{n_2 b}{2+(N+n_2-2)b}(q - q^{**}). \tag{B.5}$$

Similarly, subtracting (18) from (16) side by side yields

$$q^*(p) - q^{**} = -\frac{n_1 b}{2+(N+n_1-2)b}(p - p^{**}). \tag{B.6}$$

From (B.5) and (B.6) we obtain

$$p^*(q) - p^{**} = \frac{n_1 n_2 b^2}{[2+(N+n_1-2)b][2+(N+n_2-2)b]}(p - p^{**}),$$

and

$$q^*(p) - q^{**} = \frac{n_1 n_2 b^2}{[2+(N+n_1-2)b][2+(N+n_2-2)b]}(q - q^{**}).$$

Since $N \geqq 2$, we have $2+(N+n_1-2)b > n_1 b$, and $2+(N+n_2-2)b > n_2 b$. Then we find

$$0 < \frac{n_1 n_2 b^2}{[2+(N+n_1-2)b][2+(N+n_2-2)b]} < 1.$$

Thus, there is a sequence of the states which starts from $\omega(p', q')$, passes through $\omega(p^*(q'), q')$, through $\omega(p^*(q'), q^*(p^*(q')))$, through $\omega(p^*(q^*(p^*(q'))), q^*(p^*(q')))$ and so on. This sequence approaches to $\omega(p^*, q^*)$. If the grid of the firms' prices is sufficiently fine, this sequence converges to some limit state which is in some neighborhood of $\omega(p^{**}, q^{**})$. Then, we obtain Lemma 2.

Denote a limit state of some sequence by $\omega(p^0, q^0)$. As $\delta' \to 0$, any $\omega(p^0, q^0)$ converges to $\omega(p^{**}, q^{**})$. Then we find $C(p^{**}, q^{**}) = (v'+1)^2 - 1$, which is the number of the states other than $\omega(p^{**}, q^{**})$ given δ'. It is smaller than $C(p, q)$ for $(p, q) \neq (p^{**}, q^{**})$. Thus, we obtain Corollary 2. □

References

Alchian, A., 1950. Uncertainty, evolution and economic theory. Journal of Political Economy 58, 211–221.

Cheng, L., 1985. Comparing Bertrand and Cournot equilibria: a geometric approach. Rand Journal of Economics 16, 146–147.

Crawford, V.P., 1991. An evolutionary interpretation of Van Huyck, Battalio, and Beil's experimental results on coordination. Games and Economic Behavior 3, 83–94.

Freidlin, M.I., Wentzel, A.D., 1984. Random Perturbations of Dynamical Systems. Springer-Verlag.

Galesloot, B.M., Goyal, S., 1997. Costs of flexibility and equilibrium selection. Journal of Mathematical Economics 28, 249–264.

Hansen, R.G., Samuelson, W., 1988. Evolution in economic games. Journal of Economic Behavior and Organization 10, 315–338.

Kandori, M., Rob, R., 1995. Evolution of equilibria in the long run: a general theory and applications. Journal of Economic Theory 65, 383–414.

Kandori, M., Rob, R., 1998. Bandwagon effects and long run technology choice. Games and Economic Behavior 22, 30–60.

Kandori, M., Mailath, G., Rob, R., 1993. Learning, mutation, and long run equilibria in games. Econometrica 61, 29–56.

Maynard Smith, J., 1982. Evolution and the Theory of Games. Cambridge University Press.

Rhode, P., Stegeman, M., 2000. Non-Nash equilibria of Darwinian dynamics with applications to duopoly. International Journal of Industrial Organization, In press.

Robson, A., Vega-Redondo, F., 1996. Efficient equilibrium selection in evolutionary games with random matching. Journal of Economic Theory 70, 65–92.

Schaffer, M.E., 1988. Evolutionarily stable strategies for a finite population and a variable contest size. Journal of Theoretical Biology 132, 469–478.

Schaffer, M.E., 1989. Are profit-maximisers the best survivors? A Darwinian model of economic natural selection. Journal of Economic Behavior and Organization 12, 29–45.

Schlag, K.H., 1998. Why imitate, and if so, how? Journal of Economic Theory 78, 130–156.

Shubik, M., Levitan, R., 1980. Market Structure and Innovation. John Wiley.

Singh, N., Vives, X., 1984. Price and quantity competition in a differentiated duopoly. Rand Journal of Economics 15, 546–554.

Tanaka, Y., 1998. An evolutionary game theoretic analysis of oligopoly. The Economic Review (Iwanami Shoten) 49, 308–316, In Japanese.

Tanaka, Y., 1999a. Long run equilibria in an asymmetric oligopoly. Economic Theory 14, 705–715.

Tanaka, Y., 1999b. Stochastically Stable State in a Differentiated Oligopoly: Equivalence of Quantity Setting and Price Setting Strategies, Mimeograph.

Vega-Redondo, F., 1996. Evolution, Games and Economic Behavior. Oxford University Press.

Vega-Redondo, F., 1997. The evolution of Walrasian behavior. Econometrica 67, 375–384.

Vives, X., 1985. On the efficiency of Bertrand and Cournot equilibria with product differentiation. Journal of Economic Theory 36, 166–175.

Young, H.P., 1998. Individual Strategy and Social Structure. Princeton University Press.