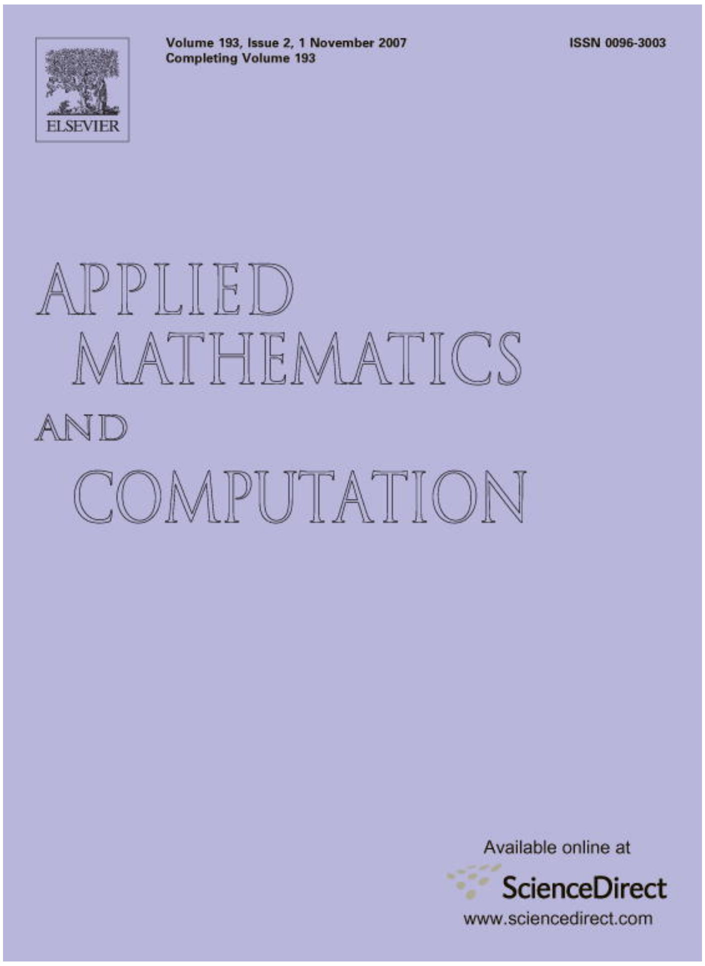
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# The Gibbard–Satterthwaite theorem of social choice theory in an infinite society and LPO (limited principle of omniscience) ☆

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**Abstract**

This paper is an attempt to examine the main theorems of social choice theory from the viewpoint of constructive mathematics. We examine the Gibbard–Satterthwaite theorem [A.F. Gibbard, Manipulation of voting schemes: a general result, Econometrica 41 (1973) 587–601; M.A. Satterthwaite, Strategyproofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions, Journal of Economic Theory 10 (1975) 187–217] in a society with an infinite number of individuals (infinite society). We will show that the theorem that any coalitionally strategy-proof social choice function may have a dictator or has no dictator in an infinite society is equivalent to LPO (limited principle of omniscience). Therefore, it is non-constructive. A dictator of a social choice function is an individual such that if he strictly prefers an alternative (denoted by $x$) to another alternative (denoted by $y$), then the social choice function chooses an alternative other than $y$. Coalitional strategy-proofness is an extension of the ordinary strategy-proofness. It requires non-manipulability for coalitions of individuals as well as for a single individual.


*Keywords:* Gibbard–Satterthwaite theorem; Infinite society; LPO (limited principle of omniscience)

## 1. Introduction

This paper is an attempt to examine the main theorems of social choice theory from the viewpoint of constructive mathematics.[1] The Gibbard–Satterthwaite theorem [5,11] shows that, with a finite number of individuals, there exists a dictator for any strategy-proof social choice function. In contrast Pazner and Wesley [10] show that in an infinite society, there exists a coalitionally strategy-proof social choice function without dictator.[2] A dictator of a social choice function is an individual such that if he strictly prefers an alternative (denoted by $x$) to another alternative (denoted by $y$), then the social choice function chooses an alternative other than $y$, and it chooses one of his most preferred alternatives. Coalitional strategy-proofness is an

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[1] In other papers, for example [14], we have studied the relationships of some other theorems of social choice theory with LPO. In another paper [15] we have examined the type 2 computability of social choice functions according to Weihrauch [17,18].

[2] Ref. [16] is a recent book that discusses social choice problems in an infinite society.

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extension of the ordinary strategy-proofness. It requires non-manipulability for coalitions of individuals as well as for a single individual.

In this paper we will show that the theorem that any coalitionally strategy-proof social choice function may have a dictator or has no dictator in an infinite society is equivalent to LPO (limited principle of omniscience). Therefore, it is non-constructive.

The omniscience principles are general statements that can be proved classically but not constructively, and are used to show that other statements do not admit constructive proofs.[3] This is done by showing that the statement implies the omniscience principle. The strongest omniscience principle is the law of excluded middle. A weaker one is the following limited principle of omniscience (abbreviated as LPO).

*Limited principle of omniscience* (LPO): Given a binary sequence $a_n$, $n \in \mathbb{N}$ (the set of positive integers), either $a_n = 0$ for all $n$ or $a_n = 1$ for some $n$.

In the next section we present the framework of this paper and some preliminary results. In Section 3 we will show the following results:

1. Any coalitionally strategy-proof social choice function may have a dictator or has no dictator, and in the latter case all co-finite sets of individuals (sets of individuals whose complements are finite) are decisive sets (Theorem 1).
2. Theorem 1 is equivalent to LPO (Theorem 2).

A decisive set for a social choice function is a set of individuals such that if individuals in the set prefer an alternative (denoted by $x$) to another alternative (denoted by $y$), then the social choice function chooses an alternative other than $y$ regardless of the preferences of other individuals.

## 2. The framework and preliminary results

There are $m(\geqslant 3)$ alternatives and a countably infinite number of individuals. $m$ is a finite positive integer. The set of individuals is denoted by $N$. The set of alternatives is denoted by $A$. $N$ and $A$ are discrete sets.[4] For each pair of elements $i, j$ of $N$ we have $i = j$ or $i \neq j$, and for each pair of elements $x, y$ of $A$ we have $x = y$ or $x \neq y$. Each subset of $N$ is a detachable set. Thus, for each individual $i$ of $N$ and each subset $I$ of $N$ we have $i \in I$ or $i \notin I$. The alternatives are represented by $x$, $y$, $z$, $w$ and so on. Denote individual $i$'s preference by $\succ_i$. We denote $x \succ_i y$ when individual $i$ prefers $x$ to $y$. Individual preferences over the alternatives are transitive weak orders, and they are characterized constructively according to Bridges [2]. About given three alternatives $x$, $y$ and $z$ individual $i$'s preference satisfies the following conditions:

1. If $x \succ_i y$, then $\neg(y \succ_i x)$.
2. If $x \succ_i y$, then for each $z \in A$ either $x \succ_i z$ or $z \succ_i y$.

Preference–indifference relation $\succsim_i$ and indifference relation $\sim_i$ are defined by

- $x \succsim_i y$ if and only if $\forall z \in A(y \succ_i z \Rightarrow x \succ_i z)$,
- $x \sim_i y$ if and only if $x \succsim_i y$ and $y \succsim_i x$.

Then, the following results are derived:

- $\neg(x \succ_i x)$.
- $x \succ_i y$ entails $x \succsim_i y$.
- The relations $\succ_i, \succsim_i$ are transitive, and $x \succsim_i y \succ_i z$ entails $x \succ_i z$.
- $x \succsim_i y$ if and only if $\neg(y \succ_i x)$.

[3] About omniscience principles we referred to [3,4,6,7].
[4] About details of the concepts of discrete set and detachable set, see [3].

As demonstrated by [2] we can not prove constructively that $x \succ_i y$ if and only if $\neg(y \succsim_i x)$.

A combination of individual preferences, which is called a *profile*, is denoted by $\mathbf{p}(=(\succ_1, \succ_2, \ldots))$, $\mathbf{p}'(=(\succ'_1, \succ'_2, \ldots))$ and so on.

We consider social choice functions which choose at least one and at most one alternative corresponding to each profile of the revealed preferences of individuals. We require that social choice functions are *coalitionally strategy-proof*. This means that any group (coalition) of individuals can not benefit by revealing preferences which are different from their true preferences, in other words, each coalition of individuals must have incentives to reveal their true preferences, and they cannot manipulate any social choice function. The coalitional strategy-proofness is an extension of the ordinary strategy-proofness which requires only non-manipulability by an individual. We also require that social choice functions are *onto*, that is, their ranges are $A$. The Gibbard–Satterthwaite theorem states that, with a finite number of individuals, there exists a dictator for any strategy-proof social choice function, or in other words there exists no social choice function which satisfies strategy-proofness and has no dictator. In contrast Pazner and Wesley [10] show that when the number of individuals in the society is infinite, there exists a coalitionally strategy-proof social choice function without dictator. A dictator of a social choice function is an individual one of whose most preferred alternatives is always chosen by the social choice function.

Now we define the following terms:

*Decisive.* If, when all individuals in a group $G$ prefer an alternative $x$ to another alternative $y$, a social choice function chooses an alternative other than $y$ regardless of the preferences of other individuals, then $G$ is *decisive* for $x$ against $y$.

*Decisive set.* If a group of individuals is decisive about every pair of alternatives for a social choice function, it is called a decisive set for the social choice function.

The meaning of the term *decisive* is similar to that of the same term used in [12] for binary social choice rules. $G$ may consist of one individual. If for a social choice function an individual is decisive about every pair of alternatives, then he is a *dictator* of the social choice function.

Further we define the following two terms.[5]

*Monotonicity.* Let $x$ and $y$ be two alternatives. Assume that at a profile $\mathbf{p}$ individuals in a group $G$ prefer $x$ to $y$, all other individuals (individuals in $N \setminus G$) prefer $y$ to $x$, and $x$ is chosen by a social choice function. If at another profile $\mathbf{p}'$ individuals in $G$ prefer $x$ to $y$, then the social choice function chooses an alternative other than $y$ regardless of the preferences of the individuals in $N \setminus G$.

*Weak Pareto principle.* If all individuals prefer $x$ to $y$, then every social choice function chooses an alternative other than $y$.

We define these terms so as to have constructive nature, and they are slightly different from the definitions in [15].

We can show the following lemmas:

**Lemma 1.** *If a social choice function satisfies coalitional strategy-proofness, then it satisfies monotonicity and weak Pareto principle.*

**Proof.** See Appendix A. □

**Lemma 2.** *Assume that a social choice function is coalitionally strategy-proof. If a group G is decisive for one alternative against another alternative, then it is a decisive set.*

**Proof.** See Appendix B. □

[5] The concept *monotonicity* is according to [1]. It is different from *strong positive association* by Muller and Satterthwaite [9] when individual preferences are weak orders (include indifference relations).

The implications of this lemma are similar to those of Lemma 3*a in [12] and Dictator Lemma in [13] for binary social choice rules.

**Lemma 3.** *Assume that a social choice function is coalitionally strategy-proof. If two groups $G$ and $G'$ are decisive sets, then their intersection $G \cap G'$ is a decisive set.*

**Proof.** See Appendix C. □

Note that $G$ and $G'$ can not be disjoint. Assume that $G$ and $G'$ are disjoint. If individuals in $G$ prefer $x$ to $y$ to all other alternatives, and individuals in $G'$ prefer $y$ to $x$ to all other alternatives, then neither $G$ nor $G'$ can be a decisive set. This lemma implies that the intersection of a finite number of decisive sets is also a decisive set.

The proofs of these lemma are almost the same as proofs of Lemmas 1–3 in [15]. But in this paper we try to present constructive proofs, in particular, the proof of Lemma 1.

## 3. Existence of coalitionally strategy-proof social choice function without dictator and LPO

Consider profiles such that one individual (denoted by $i$) prefers $x$ to $y$ to $z$ to all other alternatives, and all other individuals prefer $z$ to $x$ to $y$ to all other alternatives. Denote such a profile by $\mathbf{p}^i$. By weak Pareto principle any social choice function chooses $x$ or $z$. If a social choice function chooses $x$ at $\mathbf{p}^i$ for some $i$, then by monotonicity individual $i$ is decisive for $x$ against $z$, and by Lemma 2 he is a dictator. On the other hand, if a social choice function chooses $z$ at $\mathbf{p}^i$ for all $i \in N$, then there exists no dictator, and a group $N \setminus \{i\}$ is a decisive set for all $i \in N$. By Lemma 3 in the latter case all co-finite sets (sets of individuals whose complements are finite sets) are decisive sets. Thus, we obtain the following theorem.

**Theorem 1.** *Any coalitionally strategy-proof social choice function may have a dictator or has no dictator, and in the latter case all co-finite sets are decisive sets.*

But we can show the following theorem.

**Theorem 2.** *Theorem 1 is equivalent to LPO.*

**Proof.** We define a binary sequence $(a_i)$ as follows:

$a_i = 1$ for $i \in \mathbb{N}$ if the social choice function chooses $x$ at $\mathbf{p}^i$.
$a_i = 0$ for $i \in \mathbb{N}$ if the social choice function chooses $z$ at $\mathbf{p}^i$.

The condition of LPO for this binary sequence is as follows:
*Limited principle of omniscience* (LPO)

$$a_i = 0 \quad \text{for all } i \in \mathbb{N} \qquad \text{or} \qquad a_i = 1 \quad \text{for some } i \in \mathbb{N}.$$

From the arguments before Theorem 1 it is clearly equivalent to Theorem 1. □

## 4. Concluding remarks

We have examined the Gibbard–Satterthwaite theorem of social choice theory in an infinite society, and have shown that the theorem that any coalitionally strategy-proof social choice function may have a dictator or has no dictator in an infinite society is equivalent to LPO (limited principle of omniscience), and so it is nonconstructive. The assumption of an infinite society seems to be unrealistic. But Mihara [8] presented an interpretation of an infinite society based on a *finite* number of individuals and a countably infinite number of uncertain states.

## Appendix A. Proof of Lemma 1

We use notations in the definition of monotonicity.

1. (Monotonicity). Let $z$ be an arbitrary alternative other than $x$ and $y$. Assume that at a profile $\mathbf{p}''$ individuals in $G$ prefer $x$ to $y$ to all other alternatives, and other individuals prefer $y$ to $x$ to all other alternatives. If, when the preferences of some individuals in $G$ change from $\succ_i$ (their preferences at $\mathbf{p}$) to $\succ_i''$ (their preferences at $\mathbf{p}''$), an alternative other than $x$ is chosen by the social choice function, then they can gain benefit by revealing their preferences $\succ_i$ when their true preferences are $\succ_i''$. Thus, the social choice function continues to choose $x$ in this case. By the same logic, when the preferences of all individuals in $G$ change to their preferences at $\mathbf{p}''$, the social choice function chooses $x$. Next, if, when the preferences of some individuals in $N \setminus G$ change from $\succ_i$ to $\succ_i''$, the social choice function chooses $y$, then they can gain benefit by revealing their preferences $\succ_i''$ when their true preferences are $\succ_i$. On the other hand, if $z$ is chosen in this case, they can gain benefit by revealing their preferences $\succ_i$ when their true preferences are $\succ_i''$. Thus, $x$ must be chosen. By the same logic, when the preferences of all individuals change to their preferences at $\mathbf{p}''$, the social choice function chooses $x$. Choice of $x$ by the society never violates the coalitional strategy-proofness.
   Next, if, when the preferences of some individuals in $G$ change from $\succ_i''$ to $\succ_i'$ (their preferences at $\mathbf{p}'$), the alternative chosen by the social choice function changes directly from $x$ to $y$, then they can gain benefit by revealing their preferences $\succ_i''$ when their true preferences are $\succ_i'$. Thus, the alternative chosen by the social choice function does not directly change from $x$ to $y$ in this case. By the same logic, when the preferences of all individuals in $G$ change to their preferences at $\mathbf{p}'$, the alternative chosen by the social choice function does not directly change from $x$ to $y$. Further, if, when the preferences of some individuals in $N \setminus G$ change from $\succ_i''$ to $\succ_i'$, the alternative chosen by the social choice function changes directly from $x$ to $y$, then they can gain benefit by revealing their preference $\succ_i'$ when their true preferences are $\succ_i''$. By the same logic, when the preferences of all individuals change to their preferences at $\mathbf{p}'$, the alternative chosen by the social choice function does not directly change from $x$ to $y$.
   There is a possibility, however, that the alternative chosen by the social choice function changes from $x$ through $w(\neq x, y)$ to $y$ in transition from $\mathbf{p}''$ to $\mathbf{p}'$. If, when the preferences of some individuals change, the alternative chosen by the social choice function changes from $x$ to $w$, and further when the preferences of other some individuals (denoted by $i$) change, the alternative chosen by the social choice function changes to $y$, they have incentives to reveal their preferences $\succ_i'$ when their true preferences are $\succ_i''$ because they prefer $y$ to $w$ at $\mathbf{p}''$. Therefore, an alternative other than $y$ is chosen by the social choice function at $\mathbf{p}'$. Choice of $x$ or another alternative $w(\neq x, y)$ by the society never violates the coalitional strategy-proofness.
2. (Weak Pareto principle). Let $\mathbf{p}$ be a profile at which all individuals prefer $x$ to $y$, and $\mathbf{p}'$ be a profile at which $x$ is chosen by the social choice function. Assume that at another profile $\mathbf{p}''$ all individuals prefer $x$ to $y$ to all other alternatives. If, when the preferences of some individuals change from $\succ_i'$ to $\succ_i''$, the social choice function chooses an alternative other than $x$, then they can gain benefit by revealing their preferences $\succ_i'$ when their true preferences are $\succ_i''$. Thus, $x$ is chosen in this case. By the same logic, when the preferences of all individuals change to their preferences at $\mathbf{p}''$, $x$ is chosen. Since at $\mathbf{p}''$ and at $\mathbf{p}$ all individuals prefer $x$ to $y$, monotonicity (proved in (1)) implies that an alternative other than $y$ is chosen by the social choice function at $\mathbf{p}$.
   Choice of $x$ or another alternative $w(\neq x, y)$ by the society at $\mathbf{p}$ never violates the coalitional strategy-proofness. For example, let $w$ be an alternative other than $x$ and $y$ and assume that $\mathbf{p}$ is a profile such that all individuals prefer $w$ to $x$ to $y$ to all other alternatives, then $w$ is chosen by any social choice function. □

## Appendix B. Proof of Lemma 2

1. Case 1: There are more than three alternatives. Assume that $G$ is decisive for $x$ against $y$. Let $z$ and $w$ be given alternatives other than $x$ and $y$. Consider the following profile.
   (a) Individuals in $G$ prefer $z$ to $x$ to $y$ to $w$ to all other alternatives.

(b) Other individuals prefer $y$ to $w$ to $z$ to $x$ to all other alternatives.
By weak Pareto principle the social choice function chooses $y$ or $z$. Since $G$ is decisive for $x$ against $y$, $z$ is chosen. Then, by monotonicity the social choice function chooses an alternative other than $w$ so long as the individuals in $G$ prefer $z$ to $w$. It means that $G$ is decisive for $z$ against $w$. From this result by similar procedures we can show that $G$ is decisive for $x$ (or $y$) against $w$, for $z$ against $x$ (or $y$), and for $y$ against $x$. Since $z$ and $w$ are arbitrary, $G$ is decisive about every pair of alternatives, that is, it is a decisive set.

2. Case 2: There are only three alternatives $x$, $y$ and $z$.
Assume that $G$ is decisive for $x$ against $y$. Consider the following profile:
(a) Individuals in $G$ prefer $x$ to $y$ to $z$.
(b) Other individuals prefer $y$ to $z$ to $x$.
By weak Pareto principle the social choice function chooses $x$ or $y$. Since $G$ is decisive for $x$ against $y$, $x$ is chosen. Then, by monotonicity the social choice function chooses an alternative other than $z$ so long as the individuals in $G$ prefer $x$ to $z$. It means that $G$ is decisive for $x$ against $z$. Similarly we can show that $G$ is decisive for $z$ against $y$ considering the following profile:
(a) Individuals in $G$ prefer $z$ to $x$ to $y$.
(b) Other individuals prefer $y$ to $z$ to $x$.
By similar procedures we can show that $G$ is decisive for $y$ against $z$, for $z$ against $x$, and for $y$ against $x$. □

## Appendix C. Proof of Lemma 3

Let $x$, $y$ and $z$ be given three alternatives, and consider the following profile:

1. Individuals in $G \setminus (G \cap G')$ prefer $z$ to $x$ to $y$ to all other alternatives.
2. Individuals in $G' \setminus (G \cap G')$ prefer $y$ to $z$ to $x$ to all other alternatives.
3. Individuals in $G \cap G'$ prefer $x$ to $y$ to $z$ to all other alternatives.
4. Individuals in $N \setminus (G \cup G')$ prefer $z$ to $y$ to $x$ to all other alternatives.

Since $G$ and $G'$ are decisive sets, the social choice function chooses $x$. Only individuals in $G \cap G'$ prefer $x$ to $z$ and all other individuals prefer $z$ to $x$. Thus, by monotonicity $G \cap G'$ is decisive for $x$ against $z$. By Lemma 2 it is a decisive set. □

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