International Journal of Computational and Applied Mathematics.
ISSN 1819-4966 Volume 7, Number 1 (2012), pp. 51–56

http://www.ripublication.com/ijcam.htm

Constructive Versions of KKM Lemma and Brouwer's Fixed Point Theorem

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Abstract

It is often said that Brouwer's fixed point theorem cannot be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma. We present an alternative proof of this theorem using a constructive version of KKM (Knaster, Kuratowski and Mazurkiewicz) lemma which is proved by Sperner's lemma.

AMS subject classification: 03F65, 26E40.
Keywords: Constructive version of KKM lemma, constructive version of Brouwer's fixed point theorem, Sperner's lemma.

1. Introduction

It is often said that Brouwer's fixed point theorem cannot be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma. See [2] and [6]. We present an alternative proof of this theorem using a constructive version of KKM (Knaster, Kuratowski and Mazurkiewicz) lemma which is proved by Sperner's lemma[2].

[1]This research was partially supported by the Ministry of Education, Science, Sports and Culture of Japan, Grant-in-Aid for Scientific Research (C), 20530165.

[2]The original paper of KKM lemma is [4].

2. Sperner's lemma

Let Δ denote an n-dimensional simplex. For example, a 2-dimensional simplex is a triangle. Let partition the simplex. Figure 1 is an example of partition of a 2-dimensional simplex. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the way above mentioned, and draw the planes parallel to the faces of Δ. Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension.

Let K denote the set of small n-dimensional simplices of Δ constructed by partition. Vertices of these small simplices of K are labeled with the numbers $0, 1, 2, \dots, n$ subject to the following rules.

1. The vertices of Δ are respectively labeled with 0 to n. We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1 \dots, 0)$ with 2, $\dots$, a point $(0, \dots, 0, 1)$ with n. That is, a vertex whose k-th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k.

2. If a vertex of K is contained in an $n-1$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of one of the vertices of that face.

3. If a vertex of K is contained in an $n-2$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of one of the vertices of that face. And similarly for cases of lower dimension.

4. A vertex contained inside of Δ is labeled with an arbitrary number among 0, 1, $\dots, n$.

A small simplex of K which is labeled with the numbers $0, 1, \dots, n$ is called a *fully labeled simplex*. Sperner's lemma is stated as follows.

Lemma 2.1. [Sperner's lemma] If we label the vertices of K following the rules (1) $\sim$ (4), then there are an odd number of fully labeled simplices, and so there exists at least one fully labeled simplex.

Proof. About constructive proofs of Sperner's lemma see [3] or [5]. ■

3. Constructive version of KKM lemma

In this section we present a constructive version of KKM(Knaster, Kuratowski and Mazurkiewicz) lemma using Sperner's lemma.

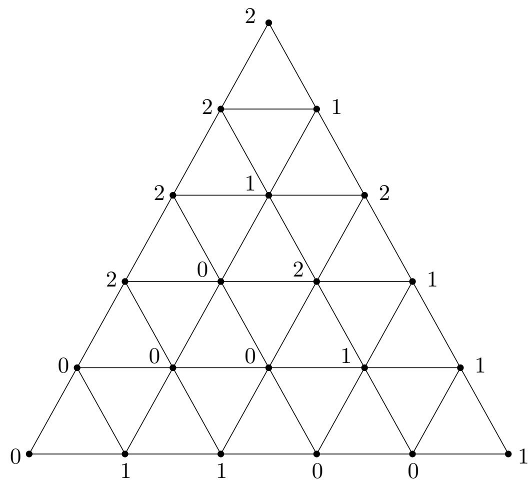

Figure 1: Partition and labeling of 2-dimensional simplex

Lemma 3.1. [constructive version of KKM Lemma] Let Δ be an n-dimensional simplex, and $p_0, p_1, \dots, p_n$ be vertices of Δ. Let Δ_i^k, $k = 0, 1, \dots, n$ be a k-dimensional face of Δ, $p_{i_0}, p_{i_1}, \dots, p_{i_k}$ be its vertices. $\{p_{i_0}, p_{i_1}, \dots, p_{i_k}\}$ is a subset of $\{p_0, p_1, \dots, p_n\}$. Let $A_0, A_1, \dots, A_n$ be non-empty subsets of Δ which satisfy the following condition[3].

$$\forall k \ \Delta_i^k \subset \bigcup_{j=0}^{k} A_{i_j}.$$

Then, for each $\varepsilon > 0$ we have $\bigcap_{i=0}^{n} V(A_i, \varepsilon) \neq \emptyset$, and we can constructively find a point p such that $p \in \bigcap_{i=0}^{n} V(A_i, \varepsilon)$, where $V(A_i, \varepsilon)$ is an ε-neighborhood of A_i.

Proof. Let K be the set of small n-dimensional simplices constructed by partition of an n-dimensional simplex Δ. The vertices $p_0, p_1, \dots, p_n$ of Δ are labeled with, respectively, $0, 1, \dots$ and n. Each vertex of all simplices of K is contained in some face of Δ. If a vertex p is contained in more than one faces of Δ, we select a face of the least dimension. Let k be its dimension and denote it by Δ_i^k. By the assumption p is contained in at least one of $A_{i_0}, A_{i_1}, \dots$ and A_{i_k}. Denote it by A_{i_j}, and label p with i_j. By the condition of this lemma i_j is the number (label) of a vertex of Δ_i^k. This labeling satisfies the condition for Sperner's lemma. Thus, there exists a fully labeled n-dimensional simplex of K. Denote the vertices of this simplex by $q_0, q_1, \dots$ and q_n. We can name them such that q_i is labeled with i. Then, each q_i is contained in A_i. If partition of Δ is sufficiently

[3]Usually in KKM lemma $A_0, A_1, \dots, A_n$ are assumed to be closed sets. But in this lemma we do not assume so.

fine, the size of this fully labeled n-dimensional simplex is also sufficiently small, and we can make all $V(A_i, \varepsilon)$'s contain this simplex. Then, this simplex is contained in the intersection of all $V(A_i, \varepsilon)$'s. Therefore, we have $\bigcap_{i=0}^{n} V(A_i, \varepsilon) \neq \emptyset$, and we can constructively find a point in this set. ■

4. Constructive version of Brouwer's fixed point theorem

First we define uniform continuity of functions and an approximate fixed point.

Definition 4.1. [Uniform continuity of functions] A function f is uniformly continuous if for each x_1, x_2 and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x_2 - x_1| < \delta$, then $|f(x_2) - f(x_1)| < \varepsilon$.

Definition 4.2. [An approximate fixed point] For each $\varepsilon > 0$ x is an approximate fixed point of f if $|x - f(x)| < \varepsilon$.

Now, using the constructive version of KKM lemma proved in the previous section, we shall show that there exists an approximate fixed point for any uniformly continuous function from an n-dimensional simplex to itself.

Theorem 4.3. [Constructive version of Brouwer's fixed point theorem] Any uniformly continuous function from an n-dimensional simplex Δ to itself has an approximate fixed point.

Proof. Assume that a function f from an n-dimensional simplex Δ to itself is uniformly continuous. Let p_0, $p_1, \dots$ and p_n be the vertices of Δ. Then a point x in Δ is represented as $x = \sum_{i=0}^{n} \lambda_i p_i$, $\sum_{i=0}^{n} \lambda_i = 1$, $\lambda_i \geq 0$. Each p_i denotes a point whose i-th component is 1. Components of $f(x)$ and x are, respectively, denoted by $f_i(x)$ and x_i. Let $\tau > 0$ be a real number, define a set A_i as follows[4],

$$A_i = \{x \in \Delta | f_i(x) < x_i + \tau\}.$$

Let x be a point contained in an $n-1$-dimensional face of Δ such that $x_i = 0$ for one i among $0, 1, 2, \dots, n$ (i-th component of its coordinates is 0)[5]. Then, from $\sum_{j=0}^{n} f_j(x) =$

[4]In constructive mathematics, for any real number x we can not prove that $x \leq 0$ or $x > 0$, that $x > 0$ or $x = 0$ or $x < 0$. Therefore, we can not define the following set.

$$B_i = \{x \in \Delta | f_i(x) \leq x_i\}.$$

But for any distinct real numbers x, y and z such that $x > z$ we can prove that $x > y$ or $y > z$. Thus, we can define a set such as our A_i. See [1] about constructive mathematics.

[5]This face of Δ is a convex hull of the vertices of Δ other than p_i. Similarly in cases below.

$\sum_{j=0}^{n} x_j = 1$ and $x_i = 0$ we obtain

$$\sum_{j=0, j\neq i}^{n} f_j(x) \leq \sum_{j=0, j\neq i}^{n} x_j.$$

Thus, there exists at least one $j\,(j \neq i)$(denote it by k) such that

$$f_k(x) < x_k + \tau.$$

Therefore, x is contained in A_k. Similarly we can show that in a face of Δ of less than $n-1$ dimension a point x such that $x_i = 0$ for some multiple i's among $0, 1, 2, \dots, n$ is contained in A_k for some k such that $x_k \neq 0$.

Because the conditions for the constructive version of KKM lemma are satisfied, we get

$$\bigcap_{i=0}^{n} V(A_i, \varepsilon) \neq \emptyset.$$

Let x be a point in $\bigcap_{i=0}^{n} V(A_i, \varepsilon)$. Then, for each i the distance between x and some x' which satisfies $f_i(x') < x'_i + \tau$ is smaller than ε. The uniform continuity of f implies that for $\tau > 0$ we can select $\varepsilon > 0$ such that when $|x - x'| < \varepsilon$ we have $\left|f(x') - f(x)\right| < \tau$. Then, we obtain

$$\left|f_i(x') - x'_i - (f_i(x) - x_i)\right| < \tau + \varepsilon.$$

Since x' satisfies $f_i(x') < x'_i + \tau$, we have

$$f_i(x) - x_i < 2\tau + \varepsilon \text{ for all } i. \tag{4.1}$$

Adding (4.1) side by side except for one i yields

$$\sum_{j=0, j\neq i}^{n} f_j(x) - \sum_{j=0, j\neq i}^{n} x_j < 2n\tau + n\varepsilon.$$

From $\sum_{j=0}^{n} f_j(x) = \sum_{j=0}^{n} x_j = 1$ we have $1 - f_i(x) < 1 - x_i + 2n\tau + n\varepsilon$, and so

$$f_i(x) - x_i > -2n\tau - n\varepsilon \tag{4.2}$$

is obtained. From (4.1) and (4.2)

$$-2n\tau - n\varepsilon < f_i(x) - x_i < 2\tau + \varepsilon.$$

Since n is finite, $2n\tau + n\varepsilon$ is a positive real number which may be arbitrarily small. Let denote anew $2n\tau + n\varepsilon$ by ε, we obtain the following result

$$\forall i \ \ |f_i(x) - x_i| < \varepsilon.$$

Further redefining $(n+1)\varepsilon$ by ε, we obtain

$$|f(x) - x| < \varepsilon.$$

Therefore, x is an approximate fixed point of $f(x)$. All points contained in $\bigcap_{i=0}^{n} V(A_i, \varepsilon)$ satisfy this relation. We have completed the proof. ■

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