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A proof of the existence of an approximate Nash equilibrium in a finite strategic game directly by Sperner's lemma: A constructive analysis[1]

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Abstract

We constructively prove the existence of an approximate Nash equilibrium in a finite strategic game directly by Sperner's lemma. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

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1. Introduction

It is often said that Brouwer's fixed point theorem can not be constructively proved.

> [5] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics a la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive. See [2] or [4]. Brouwer's fixed point theorem can be constructively, in the sense of constructive mathematics a la Bishop, proved only approximately. The existence of an exact fixed point of a function which satisfies some property of local non-constancy may be constructively proved.

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Thus, the existence of Nash equilibrium in a finite strategic game (a strategic game with a finite number of players and a finite number of pure strategies) also can not be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's theorem using Sperner's lemma. See [4] and [10]. Thus, Brouwer's fixed point theorem can be constructively proved in its constructive version, and it seems that we can prove the existence of an approximate Nash equilibrium using the constructive version of Brouwer's fixed point theorem.

Then, can we prove the existence of an approximate Nash equilibrium in a finite strategic game directly by Sperner's lemma?

We present such a proof. An approximate (or an ε-approximate) Nash equilibrium is a state such that mixed strategies chosen by all players are optimal responses each other within the range of ε.

In the next section we consider Sperner's lemma. Its constructive proof is omitted.. In Section 3 we will show the existence of an approximate Nash equilibrium in a finite strategic game by Sperner's lemma. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

2. Sperner's lemma

Let Δ denote an n-dimensional simplex. n is a finite natural number. For example, a 2-dimensional simplex is a triangle. Let partition or triangulate the simplex. Figure 1 is an example of partition (triangulation) of a 2-dimensional simplex. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ. m is a finite natural number. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is an 2-dimensional simplex, and so it is partitioned into m^2 triangles in the way above mentioned, and draw the planes parallel to the faces of Δ. Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension.

Let K denote the set of small n-dimensional simplices of Δ constructed by partition. Vertices of these small simplices of K are labeled with the numbers $0, 1, 2, \dots, n$ subject to the following rules.

1. The vertices of Δ are respectively labeled with 0 to n. We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1 \dots, 0)$ with $2, \dots,$ a point $(0, \dots, 0, 1)$ with n. That is, a vertex whose k-th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k.

2. If a vertex of K is contained in an $n-1$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of one of the vertices of that face.

3. If a vertex of K is contained in an $n-2$-dimensional face of Δ, then this vertex is

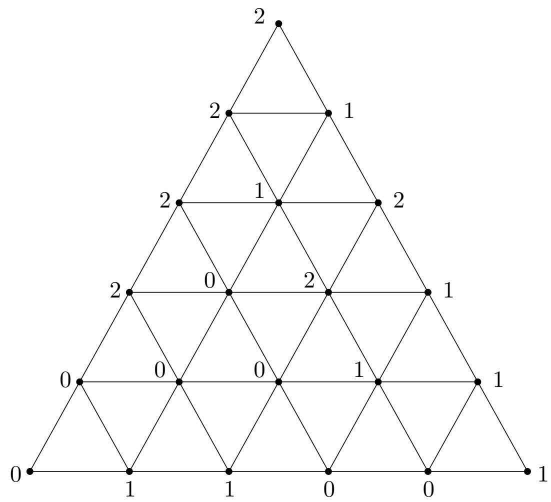

Figure 1: Partition and labeling of 2-dimensional simplex

labeled with some number which is the same as the number of one of the vertices of that face. And similarly for cases of lower dimension.

4. A vertex contained inside of Δ is labeled with an arbitrary number among 0, 1, $\dots, n$.

A small simplex of K which is labeled with the numbers 0, 1, $\dots$, n is called a *fully labeled simplex*. Sperner's lemma is stated as follows.

Lemma 2.1. [Sperner's lemma] If we label the vertices of K following the rules $1 \sim 4$, then there are an odd number of fully labeled simplices, and so there exists at least one fully labeled simplex.

Proof. About constructive proofs of Sperner's lemma see [7] or [8]. ■

Since n and partition of Δ are finite, the number of small simplices constructed by partition is also finite. Thus, we can constructively find a fully labeled n-dimensional simplex of K through finite steps.

3. Approximate Nash equilibrium

Using Sperner's lemma we look at the problem of the existence of an approximate Nash equilibrium in a finite strategic game according to [6]. Let $\varepsilon > 0$ be a small number. If a strategy of a player is an optimal response to strategies of other players within the range of ε, we call it an *approximate optimal response*, and call a state where all players choose their approximate optimal responses each other an *approximate Nash equilibrium*.

Consider an n-players strategic game with m pure strategies for each player. n and m are finite natural numbers not smaller than 2. Let S_i be the set of pure strategies for player i, and denote his each pure strategy by s_{ij}. His mixed strategy is defined as a probability distribution over S_i, and is denoted by p_i. Let p_{ij} be a probability that player i chooses s_{ij}, then $\sum_{j=1}^{m} p_{ij} = 1$ for all i. A combination of mixed strategies of all players is called a *profile*, and is denoted by $\mathbf{p}$. Let $\pi_i(\mathbf{p})$ be the expected payoff of player i at profile $\mathbf{p}$, and $\pi_i(s_{ij}, \mathbf{p}_{-i})$ be his payoff when he chooses a strategy s_{ij} at that profile, where $\mathbf{p}_{-i}$ denotes a combination of mixed strategies of players other than i at profile $\mathbf{p}$. $\pi_i(\mathbf{p})$ is written as follows,

$$\pi_i(\mathbf{p}) = \sum_{\{j: p_{ij}>0\}} p_{ij}\pi_i(s_{ij}, \mathbf{p}_{-i})$$

We assume that the values of payoffs of all players are finite. Then, since pure strategies are finite and expected payoffs are linear functions about probability distributions over the sets of pure strategies of all players, $\pi_i(\mathbf{p})$ is uniformly continuous about $\mathbf{p}$. Let $\tilde{S}_i$ be the set of player i's approximate optimal responses to $\mathbf{p}_{-i}$. It includes pure strategies which satisfy the following condition.

$$\tilde{S}_i = \{s_{ij} | \text{for all } s'_{ij} \in S_i,\ \pi_i(s_{ij}, \mathbf{p}_{-i}) > \pi_i(s'_{ij}, \mathbf{p}_{-i}) - \varepsilon\}.$$

Mixed strategies which assign positive probabilities to only pure strategies satisfying this condition are also approximate optimal responses for player i. Each player chooses one of his approximate optimal responses corresponding to strategies of other players. Now we define the following function.

$$\psi_{ij}(\mathbf{p}) = \frac{p_{ij} + \max(\pi_i(s_{ij}, \mathbf{p}_{-i}) - \pi_i(\mathbf{p}), 0)}{1 + \sum_{k=1}^{m} \max(\pi_i(s_{ik}, \mathbf{p}_{-i}) - \pi_i(\mathbf{p}), 0)},$$

where $\sum_{j=1}^{m} \varphi_{ij} = 1$ for all i. Let $\psi_i(\mathbf{p}) = (\psi_{i_1}, \psi_{i_2}, \dots$ and $\psi_{i_m})$, $\psi(\mathbf{p}) = (\psi_1, \psi_2, \cdots, \psi_n)$. Since each ψ_i is an m-dimensional vector such that the values of its components are between 0 and 1, and the sum of its components is 1, it represents a point on an $m-1$-dimensional simplex. The vertices of the simplex correspond to pure strategies. $\psi(\mathbf{p})$ is a combination of vectors ψ_i's. It is a vector such that its components consist of components of $\psi_i(\mathbf{p})$ of all players. Thus, it is a vector with $n \times m$ components, but since the number of independent components is $n(m-1)$, the range of $\psi(\mathbf{p})$ is the n-times product of $m-1$-dimensional simplices. It is convex, and homeomorphic to $n(m-1)$-dimensional simplex. Let denote the n-times product of $m-1$-dimensional simplices by $(\Delta^{m-1})^n$ and an $n(m-1)$-dimensional simplex by $\Delta^{n(m-1)}$. $\mathbf{p} = (p_1, p_2, \dots, p_n)$ is also a vector with $n \times m$ components, and the number of its independent components is $n(m-1)$. Thus, the domain of $\psi(\mathbf{p})$ is the same set as the range of $\psi(\mathbf{p})$, and a uniformly continuous function from $n(m-1)$-dimensional simplex to itself corresponds one to one to a uniformly continuous function from the

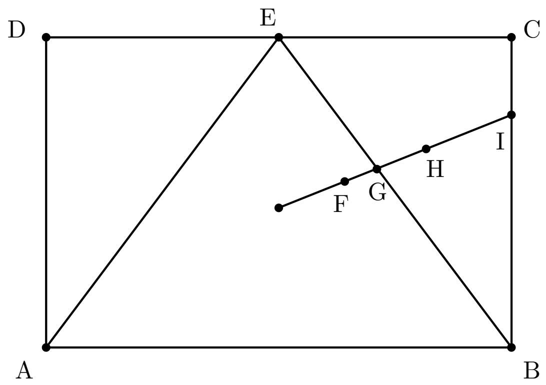

Figure 2: Homeomorphism between $(\Delta^{m-1})^n$ and $\Delta^{n(m-1)}$

domain of $\psi(\mathbf{p})$ to its range. The relation between $(\Delta^{m-1})^n$ and $\Delta^{n(m-1)}$ in the case where $n = m = 2$ is illustrated in Figure 2. F corresponds to H, and G corresponds to I.

Let $\mathbf{q}$ be a point on $\Delta^{n(m-1)}$ and $\mathbf{p} = \eta(\mathbf{q})$ be a uniformly continuous homeomorphism between $(\Delta^{m-1})^n$ and $\Delta^{n(m-1)}$. Then,

$$\zeta(\mathbf{q}) = \psi \circ \eta(\mathbf{q})$$

is a uniformly continuous function from $\Delta^{n(m-1)}$ to itself.

Now we show the following theorem.

Theorem 3.1. In any finite strategic game there exists an approximate Nash equilibrium.

Proof.

1. First we show that we can partition $\Delta^{n(m-1)}$, which is the domain and range of $\zeta(\mathbf{q})$, so that the conditions for Sperner's lemma are satisfied. We partition $\Delta^{n(m-1)}$ according to the method in Sperner's lemma, and label the vertices of simplices constructed by partition of $\Delta^{n(m-1)}$. It is important how to label the vertices contained in the faces of $\Delta^{n(m-1)}$. Let K be the set of small simplices constructed by partition of $\Delta^{n(m-1)}$, $\mathbf{q} = (q_0, q_1, \dots, q_{n(m-1)})$ be a vertex of a simplex of K, and denote the i-th component of $\zeta(\mathbf{q})$ by ζ_i. Then, we label a vertex $\mathbf{q}$ according to the following rule,

   $$\text{If } q_k > \zeta_k \text{ or } q_k + \tau > \zeta_k, \text{ we label } \mathbf{q} \text{ with } k,$$

   where $\tau > 0$. If there are multiple k's which satisfy this condition, we label $\mathbf{q}$ conveniently for the conditions for Sperner's lemma to be satisfied. We do not randomly label the vertices.

   For example, let $\mathbf{q}$ be a point contained in an $n(m-1)-1$-dimensional face of $\Delta^{n(m-1)}$ such that $q_i = 0$ for one i among $0, 1, 2, \dots, n(m-1)$ (its i-th coordinate is 0). With $\tau > 0$, $\zeta_i > 0$ or $\zeta_i < \tau$[2]. When $\zeta_i > 0$, from $\sum_{j=0}^{n(m-1)} q_j = 1$, $\sum_{j=0}^{n(m-1)} \zeta_j = 1$ and $q_i = 0$ we have

$$\sum_{j=0, j\neq i}^{n(m-1)} q_j > \sum_{j=0, j\neq i}^{n(m-1)} \zeta_j.$$

Then, for at least one j (denote it by k) we have $q_k > \zeta_k$, and we label $\mathbf{q}$ with k, where k is one of the numbers which satisfy $q_k > \zeta_k$. Since $\zeta_i > q_i$, i does not satisfy this condition. When $\zeta_i < \tau$, $q_i = 0$ implies $\sum_{j=0, j\neq i}^{n(m-1)} q_j = 1$. Since $\sum_{j=0, j\neq i}^{n(m-1)} \zeta_j \leq 1$, we obtain

$$\sum_{j=0, j\neq i}^{n(m-1)} q_j \geq \sum_{j=0, j\neq i}^{n(m-1)} \zeta_j.$$

Then, for $\tau > 0$ we have

$$\sum_{j=0, j\neq i}^{n(m-1)} (q_j + \tau) > \sum_{j=0, j\neq i}^{n(m-1)} \zeta_j.$$

There is at least one $j (\neq i)$ which satisfies $q_j + \tau > \zeta_j$. Denote it by k, and we label $\mathbf{q}$ with k. k is one of the numbers other than i such that $q_k + \tau > \zeta_k$ is satisfied. i itself satisfies this condition ($q_i + \tau > \zeta_i$). But, since there is a number other than i which satisfies this condition, we can select a number other than i. We have proved that we can label the vertices contained in an $n(m-1)-1$-dimensional face of $\Delta^{n(m-1)}$ such that $q_i = 0$ for one i among $0, 1, 2, \ldots, n(m-1)$ with the numbers other than i. By similar procedures we can show that we can label the vertices contained in an $n(m-1)-2$-dimensional face of $\Delta^{n(m-1)}$ such that $q_i = 0$ for two i's among $0, 1, 2, \ldots, n(m-1)$ with the numbers other than those i's, and so on.

[2]In constructive mathematics for any real number x we can not prove that $x \geq 0$ or $x < 0$, that $x > 0$ or $x = 0$ or $x < 0$. But for any distinct real numbers x, y and z such that $x > z$ we can prove that $x > y$ or $y > z$. See [1], [2] and [3] about constructive mathematics.

Consider the case where $q_i = q_{i+1} = 0$. By similar procedures we see that when $\zeta_i > 0$ or $\zeta_{i+1} > 0$,

$$\sum_{j=0, j\neq i,i+1}^{n(m-1)} q_j > \sum_{j=0, j\neq i,i+1}^{n(m-1)} \zeta_j.$$

Then, for at least one j(denote it by k) we have $q_k > \zeta_k$, and we label $\mathbf{q}$ with k. On the other hand, when $\zeta_i < \tau$ and $\zeta_{i+1} < \tau$, we have

$$\sum_{j=0, j\neq i,i+1}^{n(m-1)} q_j \geq \sum_{j=0, j\neq i,i+1}^{n(m-1)} \zeta_j.$$

Then, for $\tau > 0$ we have

$$\sum_{j=0, j\neq i,i+1}^{n(m-1)} (q_j + \tau) > \sum_{j=0, j\neq i,i+1}^{n(m-1)} \zeta_j.$$

Thus, there is at least one $j(\neq i, i+1)$ which satisfies $q_j + \tau > \zeta_j$. Denote it by k, and we label $\mathbf{q}$ with k.

Next consider the case where $q_i = 0$ for all i other than $n(m-1)$. If for some i $\zeta_i > 0$, then we have $q_{n(m-1)} > \zeta_{n(m-1)}$, and label $\mathbf{q}$ with $n(m-1)$. On the other hand, if $\zeta_j < \tau$ for all $j \neq n(m-1)$, then we obtain $q_{n(m-1)} \geq \zeta_{n(m-1)}$. It implies $q_{n(m-1)} + \tau > \zeta_{n(m-1)}$. Thus, we can label $\mathbf{q}$ with $n(m-1)$.

Therefore, the conditions for Sperner's lemma are satisfied, and there exists an odd number of fully labeled simplices in K.

2. Suppose that we partition $\Delta^{n(m-1)}$ sufficiently fine so that the distance between any pair of the vertices of small simplices is sufficiently small. Let $\mathbf{q}^0$, $\mathbf{q}^1, \dots$ and $\mathbf{q}^{n(m-1)}$ be the vertices of a fully labeled simplex. We name these vertices so that $\mathbf{q}^0, \mathbf{q}^1, \dots, \mathbf{q}^{n(m-1)}$ are labeled, respectively, with $0, 1, \dots, n(m-1)$. The values of ζ at theses vertices are $\zeta(\mathbf{q}^0)$, $\zeta(\mathbf{q}^1), \dots$ and $\zeta(\mathbf{q}^{n(m-1)})$. The i-th components of $\mathbf{q}^0$ and $\zeta(\mathbf{q}^0)$ are denoted by $\mathbf{q}_i^0$ and $\zeta(\mathbf{q}^0)_i$, and so on.

   From the uniform continuity of ζ we can select $\delta > 0$ so that when the distance between $\mathbf{q}^0$ and $\mathbf{q}^1$ ($|\mathbf{q}^0 - \mathbf{q}^1|$) is smaller than δ, the distance between $\zeta(\mathbf{q}^0)$ and $\zeta(\mathbf{q}^1)$ ($|\zeta(\mathbf{q}^0) - \zeta(\mathbf{q}^1)|$) is smaller than ε. We can make δ satisfying $\delta < \varepsilon$[3]. Suppose $\tau > 0$. About $\mathbf{q}^0$, from the labeling rules we have $\mathbf{q}_0^0 + \tau > \zeta(\mathbf{q}^0)_0$. About $\mathbf{q}^1$, also from the labeling rules we have $\mathbf{q}_1^1 + \tau > \zeta(\mathbf{q}^1)_1$ which implies

[3]For example, for $\delta < 1$ and $\varepsilon < 1$, if when $|\mathbf{q}^0 - \mathbf{q}^1| < \delta$ we have $|\zeta(\mathbf{q}^0) - \zeta(\mathbf{q}^1)| < \varepsilon$, then we have $|\zeta(\mathbf{q}^0) - \zeta(\mathbf{q}^1)| < \varepsilon$ also when $|\mathbf{q}^0 - \mathbf{q}^1| < \delta\varepsilon < \varepsilon$.

$\mathbf{q}_1^1 > \zeta(\mathbf{q}^1)_1 - \tau$. By the uniform continuity of ζ, $|\mathbf{q}^0 - \mathbf{q}^1| < \delta$ implies $|\zeta(\mathbf{q}^0) - \zeta(\mathbf{q}^1)| < \varepsilon$, which means $\zeta(\mathbf{q}^1)_1 > \zeta(\mathbf{q}^0)_1 - \varepsilon$. On the other hand, $|\mathbf{q}^0 - \mathbf{q}^1| < \delta$ means $\mathbf{q}_1^0 > \mathbf{q}_1^1 - \delta$. Thus, from

$$\mathbf{q}_1^0 > \mathbf{q}_1^1 - \delta,\ \mathbf{q}_1^1 > \zeta(\mathbf{q}^1)_1 - \tau,\ \zeta(\mathbf{q}^1)_1 > \zeta(\mathbf{q}^0)_1 - \varepsilon$$

we obtain

$$\mathbf{q}_1^0 > \zeta(\mathbf{q}^0)_1 - \delta - \varepsilon - \tau > \zeta(\mathbf{q}^0)_1 - 2\varepsilon - \tau$$

By similar arguments, for each i other than 0, we obtain

$$\mathbf{q}_i^0 > \zeta(\mathbf{q}^0)_i - 2\varepsilon - \tau. \tag{3.1}$$

For $i = 0$ we have $\mathbf{q}_0^0 + \tau > \zeta(\mathbf{q}^0)_0$. Then,

$$\mathbf{q}_0^0 > \zeta(\mathbf{q}^0)_0 - \tau \tag{3.2}$$

Adding (3.1) and (3.2) side by side except for some i (denote it by k) other than 0 yields

$$\sum_{j=0, j\neq k}^{n(m-1)} \mathbf{q}_j^0 > \sum_{j=0, j\neq k}^{n(m-1)} \zeta(\mathbf{q}^0)_j - 2[(n(m-1)-1]\varepsilon - n(m-1)\tau.$$

From $\sum_{j=0}^{n(m-1)} \mathbf{q}_j^0 = 1$, $\sum_{j=0}^{n(m-1)} \zeta(\mathbf{q}^0)_j = 1$ we have $1 - \mathbf{q}_k^0 > 1 - \zeta(\mathbf{q}^0)_k - 2[(n(m-1)-1]\varepsilon - n(m-1)\tau$, which is rewritten as

$$\mathbf{q}_k^0 < \zeta(\mathbf{q}^0)_k + 2[(n(m-1)-1]\varepsilon + n(m-1)\tau.$$

Since (3.1) implies $\mathbf{q}_k^0 > \zeta(\mathbf{q}^0)_k - 2\varepsilon - \tau$, we have

$$\zeta(\mathbf{q}^0)_k - 2\varepsilon - \tau < \mathbf{q}_k^0 < \zeta(\mathbf{q}^0)_k + 2[(n(m-1)-1]\varepsilon + n(m-1)\tau.$$

Thus,

$$|\mathbf{q}_k^0 - \zeta(\mathbf{q}^0)_k| < 2[(n(m-1)-1]\varepsilon + n(m-1)\tau \tag{3.3}$$

is derived. On the other hand, adding (3.1) from 1 to $n(m-1)$ yields

$$\sum_{j=1}^{n(m-1)} \mathbf{q}_j^0 > \sum_{j=1}^{n(m-1)} \zeta(\mathbf{q}^0)_j - 2n(m-1)\varepsilon - n(m-1)\tau.$$

From $\sum_{j=0}^{n(m-1)} \mathbf{q}_j^0 = 1$, $\sum_{j=0}^{n(m-1)} \zeta(\mathbf{q}^0)_j = 1$ we have

$$1 - \mathbf{q}_0^0 > 1 - \zeta(\mathbf{q}^0)_0 - 2n(m-1)\varepsilon - n(m-1)\tau. \tag{3.4}$$

Then, from (3.2) and (3.4) we get

$$|\mathbf{q}_0^0 - \zeta(\mathbf{q}^0)_0| < 2n(m-1)\varepsilon + n(m-1)\tau. \tag{3.5}$$

Since n and m are finite, and ε and τ are positive real numbers which are small, $2n(m-1)\varepsilon + n(m-1)\tau$ and $2[(n(m-1)-1]\varepsilon + n(m-1)\tau$ are also small. Redefining $2n(m-1)\varepsilon + n(m-1)\tau$ by ε, (3.3) and (3.5) yield the following result,

$$|\mathbf{q}_i^0 - \zeta(\mathbf{q}_i^0)| < \varepsilon \text{ for all } i. \tag{3.6}$$

All points contained in the fully labeled simplex of K satisfy this relation.

3. Denote a point which satisfies (3.6) by $\tilde{\mathbf{q}}$. Let $\tilde{\mathbf{p}} = \eta(\tilde{\mathbf{q}}) = (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n)$. Denote the components of $\tilde{p}_i$ by $\tilde{p}_{ij}$. Since η and ψ are uniformly continuous,

$$|\psi_{ij} - \tilde{p}_{ij}| < \lambda, \text{ for all } i \text{ and } j,$$

for some $\lambda > 0$. If ε is small, λ is also small. From the definition of ψ_{ij}, we obtain

$$\left|\frac{\tilde{p}_{ij} + \max(\pi_i(s_{ij}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0)}{1 + \sum_{k=1}^m \max(\pi_i(s_{ik}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0)} - \tilde{p}_{ij}\right|$$
$$= \left|\frac{\max(\pi_i(s_{ij}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0) - \tilde{p}_{ij}\sum_{k=1}^m \max(\pi_i(s_{ik}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0)}{1 + \sum_{k=1}^m \max(\pi_i(s_{ik}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0)}\right| < \lambda.$$

Let $\gamma = \sum_{k=1}^{m} \max(\pi_i(s_{ik}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0)$, then

$$\left|\frac{\max(\pi_i(s_{ij}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0) - \gamma\tilde{p}_{ij}}{1+\gamma}\right| < \lambda,$$

or equivalently

$$\left|\max(\pi_i(s_{ij}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0) - \gamma\tilde{p}_{ij}\right| < (1+\gamma)\lambda$$

is derived, where $\tilde{\mathbf{p}}_{-i}$ denotes a combination of mixed strategies of players other than i at profile $\tilde{\mathbf{p}}$. This implies

$$-(1+\gamma)\lambda + \gamma\tilde{p}_{ij} < \max(\pi_i(s_{ij}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0) < (1+\gamma)\lambda + \gamma\tilde{p}_{ij}.$$

Since $\pi_i(\tilde{\mathbf{p}}) = \sum_{\{j:\tilde{p}_{ij}>0\}} \tilde{p}_{ij}\pi_i(s_{ij}, \tilde{\mathbf{p}}_i)$, it is impossible that $\max(\pi_i(s_{ij}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0) = \pi_i(s_{ij}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}) > 0$ for all j satisfying $\tilde{p}_{ij} > 0$. Thus, γ as well as λ is a small real number. Redefining $(1+\gamma)\lambda + \gamma\tilde{p}_{ij}$ by ε, $\max(\pi_i(s_{ij}, \tilde{\mathbf{p}}_{-i}) - \pi_i(\tilde{\mathbf{p}}), 0) < \varepsilon$ holds for all s_{ij}'s whether $\tilde{p}_{ij} > 0$ or not, and it holds for all players. That is, strategies of all players in $\tilde{\mathbf{p}}$ are optimal responses each other within the range of ε. Thus, they are approximate optimal responses, and a state where all players choose these strategies is an approximate Nash equilibrium. ■

4. Concluding Remark

In this paper we have presented a proof of the existence of an approximate Nash equilibrium in a finite strategic game directly by Sperner's lemma from a point of view of constructive mathematics. In another paper, ([9]), we apply this method to prove the existence of an approximate equilibrium in a competitive economy.

References

[1] E. Bishop and D. Bridges, *Constructive Analysis*, Springer, 1985.

[2] D. Bridges and F. Richman, *Varieties of Constructive Mathematics*, Cambridge University Press, 1987.

[3] D. Bridges and L. Vîţă, *Techniques of Constructive Mathematics*, Springer, 2006.

[4] D. van Dalen, "Brouwer's ε-fixed point from Sperner's lemma", *Theoretical Computer Science*, http://dx.doi.org/10.1016/j.tcs.2011.04.002, in press 2011.

[5] R. B. Kellogg, T. Y. Li and J. Yorke, "A constructive proof of Brouwer fixed-point theorem and computational results", *SIAM Journal on Numerical Analysis*, 13:473–483, 1976.

[6] J. Nash, "Non-cooperative games", *Annals of Mathematics, Second Series*, 54:286–295, 1951.

[7] F. E. Su, "Rental harmony: Sperner's lemma for fair devision", *American Mathematical Monthly*, 106:930–942, 1999.

[8] Y. Tanaka, "A proof of the existence of approximate core in NTU game directly by Sperner's lemma: A constructive analysis", *Mathematics Applied in Science and Technology*, Research India Publications, in press 2011.

[9] Y. Tanaka, "Equivalence between the existence of an approximate equilibrium in a competitive economy and Sperner's lemma: A constructive analysis", *ISRN Applied Mathematics*, vol. 2011, Article ID 384625, pp. 1–15, Hindawi Publishing Corporation, 2011.

[10] W. Veldman, "Brouwer's approximate fixed point theorem is equivalent to Brouwer's fan theorem", in *Logicism, Intuitionism and Formalism*, edited by Lindström, S., Palmgren, E., Segerberg, K. and Stoltenberg-Hansen, Springer, 2009.