

Proof of constructive version of Kakutani's fixed point theorem directly by Sperner's lemma and approximate mini-max theorem: A constructive analysis¹

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Abstract

It is often said that Brouwer's fixed point theorem can not be constructively proved. Therefore, Kakutani's fixed point theorem also can not be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma. Thus, Brouwer's fixed point theorem can be constructively proved in its constructive version. It seems that Kakutani's fixed point theorem for compact and convex valued multi-functions (multi-valued functions or correspondences) with closed graph also can be constructively proved in its constructive version using that of Brouwer's theorem. Then, can we prove a constructive version of Kakutani's fixed point theorem directly by Sperner's lemma? We present such a proof, and we present an approximate version of the mini-max theorem for a zero-sum game by the constructive version of Kakutani's theorem. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

AMS subject classification: 03F65, 91A10.

Keywords: Sperner's Lemma, constructive version of Kakutani's fixed point theorem, zero-sum game, mini-max theorem.

¹This research was partially supported by the Ministry of Education, Science, Sports and Culture of Japan, Grant-in-Aid for Scientific Research (C), 20530165, and the Special Costs for Graduate Schools of the Special Expenses for Hitech Promotion by the Ministry of Education, Science, Sports and Culture of Japan in 2011.

1. Introduction

It is often said that Brouwer's fixed point theorem can not be constructively proved.

[5] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics a la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive. See [2] or [4]. Brouwer's fixed point theorem can be constructively, in the sense of constructive mathematics a la Bishop, proved only approximately. The existence of an exact fixed point of a function which satisfies some property of local non-constancy may be constructively proved.

Therefore, Kakutani's fixed point theorem also can not be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma. Thus, Brouwer's fixed point theorem can be constructively proved in its constructive version. See [4] and [7]. It seems that Kakutani's fixed point theorem for compact and convex valued multi-functions (multi-valued functions or correspondences) with closed graph also can be constructively proved in its constructive version using that of Brouwer's theorem.

Then, can we prove a constructive version of Kakutani's fixed point directly by Sperner's lemma?

We present such a proof, and we also present an approximate version of the mini-max theorem for a zero-sum game by the constructive version of Kakutani's theorem. We consider a uniform version of the closed graph property of multi-functions, and call such a multi-function a *multi-function with uniformly closed graph*, or say that a multi-function uniformly has a closed graph. The constructive version of Kakutani's fixed point theorem states that for any compact (which means totally boundedness with completeness in constructive mathematics) and convex valued multi-function with uniformly closed graph from an n -dimensional simplex to the set of its inhabited (nonempty) subsets, there is an approximate fixed point. An approximate fixed point \mathbf{p}^* of a multi-function F for each $\varepsilon > 0$ is a point which satisfies $|\mathbf{p}^* - F(\mathbf{p}^*)| < \varepsilon(|\mathbf{p}^* - \mathbf{p}| < \varepsilon$ for some $\mathbf{p} \in F(\mathbf{p}^*)$). \mathbf{p}^* depends on ε .

An approximate version of the mini-max theorem for a zero-sum game states that the value of a two person zero-sum game is determined within the range of ε , and we can constructively find the strategies of players which realize this approximate value of the game.

In the next section we prove Sperner's lemma using the Handshaking lemma of graph theory. This proof is a standard constructive proof. In Section 3 we prove a constructive version of Kakutani's fixed point theorem using Sperner's lemma. In Section 4 we present an approximate version of the mini-max theorem. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

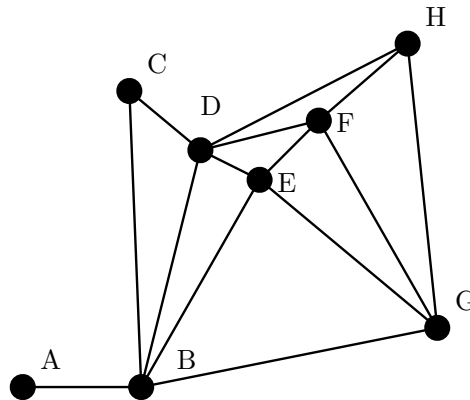


Figure 1: Example of graph

2. Sperner's lemma

To prove Sperner's lemma we use the following simple result of graph theory, Handshaking lemma². A *graph* refers to a collection of vertices and a collection of edges that connect pairs of vertices. Each graph may be undirected or directed. Figure 1 is an example of an undirected graph. Degree of a vertex of a graph is defined to be the number of edges incident to the vertex, with loops counted twice. Each vertex has odd degree or even degree. Let v denote a vertex and V denote the set of all vertices.

Lemma 2.1. [Handshaking lemma] Every undirected graph contains an even number of vertices of odd degree. That is, the number of vertices that have an odd number of incident edges must be even.

It is a simple lemma. But for completeness of arguments we provide a proof.

Proof. Prove this lemma by double counting. Let $d(v)$ be the degree of vertex v . The number of vertex-edge incidences in the graph may be counted in two different ways: by summing the degrees of the vertices, or by counting two incidences for every edge. Therefore

$$\sum_{v \in V} d(v) = 2e,$$

where e is the number of edges in the graph. The sum of the degrees of the vertices is therefore an even number. It could happen if and only if an even number of the vertices had odd degree. ■

Let Δ denote an n -dimensional simplex. n is a finite natural number. For example, a 2-dimensional simplex is a triangle. Let partition or triangulate the simplex. Figure 2 is an example of partition (triangulation) of a 2-dimensional simplex. In a 2-dimensional

²For another constructive proof of Sperner's lemma, see [6].

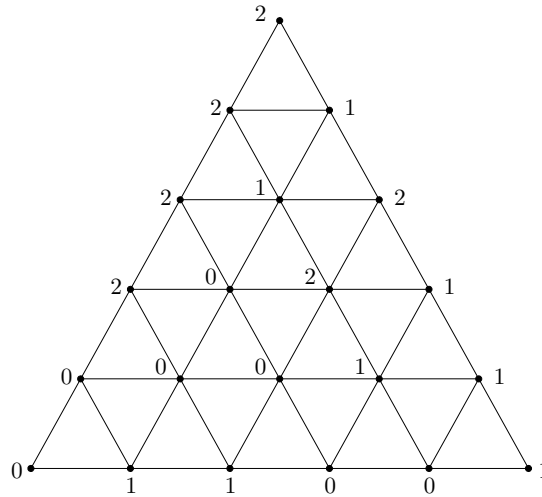


Figure 2: Partition and labeling of 2-dimensional simplex

case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ . m is a finite natural number. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the way above mentioned, and draw the planes parallel to the faces of Δ . Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension.

Let K denote the set of small n -dimensional simplices of Δ constructed by partition. Vertices of these small simplices of K are labeled with the numbers $0, 1, 2, \dots, n$ subject to the following rules.

1. The vertices of Δ are respectively labeled with 0 to n . We label a point $(1, 0, \dots, 0)$ with 0 , a point $(0, 1, 0, \dots, 0)$ with 1 , a point $(0, 0, 1, \dots, 0)$ with $2, \dots$, a point $(0, \dots, 0, 1)$ with n . That is, a vertex whose k -th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k .
2. If a vertex of K is contained in an $n - 1$ -dimensional face of Δ , then this vertex is labeled with some number which is the same as the number of one of the vertices of that face.
3. If a vertex of K is contained in an $n - 2$ -dimensional face of Δ , then this vertex is labeled with some number which is the same as the number of one of the vertices of that face. And similarly for cases of lower dimension.
4. A vertex contained inside of Δ is labeled with an arbitrary number among $0, 1, \dots, n$.

A small simplex of K which is labeled with the numbers $0, 1, \dots, n$ is called a *fully labeled simplex*. Now let us prove Sperner's lemma.

Lemma 2.2. [Sperner's lemma] If we label the vertices of K following the rules (1) ~ (4), then there are an odd number of fully labeled simplices, and so there exists at least one fully labeled simplex.

Proof. See Appendix 5. ■

Since n and partition of Δ are finite, the number of small simplices constructed by partition is also finite. Thus, we can constructively find a fully labeled n -dimensional simplex of K through finite steps.

3. Constructive version of Kakutani's fixed point theorem

Consider a multi-function F from an n -dimensional simplex Δ to the set of its inhabited (nonempty) subsets. We assume that $F(\mathbf{p})$ for $\mathbf{p} \in \Delta$ is a compact and convex set. The classical version of Kakutani's fixed point theorem states that if a compact and convex valued multi-function from Δ to the set of its inhabited subsets has a closed graph, it has a fixed point.

A graph of a multi-function F from Δ to the set of its inhabited subsets is

$$G(F) = \cup_{\mathbf{p} \in \Delta} \{\mathbf{p}\} \times F(\mathbf{p}).$$

If $G(F)$ is a closed set, we say that F has a closed graph. It implies the following fact.

Consider sequences $(\mathbf{p}_n)_{n \geq 1}$ and $(\mathbf{q}_n)_{n \geq 1}$ such that $\mathbf{q}_n \in F(\mathbf{p}_n)$. If $\mathbf{p}_n \longrightarrow \mathbf{p}$ and $\mathbf{q}_n \longrightarrow \mathbf{q}$, then $\mathbf{q} \in F(\mathbf{p})$.

According to [3] this means

If for each neighborhood $U(\mathbf{p}, \varepsilon)$ of \mathbf{p} there exists n_0 such that $\mathbf{p}_n \in U(\mathbf{p}, \varepsilon)$ when $n \geq n_0$, then for the union of neighborhoods $\cup_{\mathbf{q} \in F(\mathbf{p})} V(\mathbf{q}, \varepsilon)$ of points in $F(\mathbf{p})$ there exists n'_0 such that $\mathbf{q}_n \in \cup_{\mathbf{q} \in F(\mathbf{p})} V(\mathbf{q}, \varepsilon)$ when $n \geq n'_0$.

Further we consider a uniform version of this property for multi-functions, and call such a multi-function a *multi-function with uniformly closed graph*, or say that a multi-function uniformly has a closed graph. It means that n_0 and n'_0 depend on only ε not on \mathbf{p} .

We define an approximate fixed point of a multi-function as follows.

Definition 3.1. [Approximate fixed point of multi-function] For each $\varepsilon > 0$ \mathbf{p}^* is an approximate fixed point of a multi-function F from Δ to the set of its inhabited subsets if

$$|\mathbf{p}^* - F(\mathbf{p}^*)| < \varepsilon.$$

\mathbf{p}^* depends on ε . Now, we prove the following theorem.

Theorem 3.2. [Constructive version of Kakutani's fixed point theorem] Any compact and convex valued multi-function with uniformly closed graph from an n -dimensional simplex Δ to the sets of its inhabited subsets has an approximate fixed point.

Proof.

1. We partition Δ according to the method in the proof of Sperner's lemma, and label the vertices of simplices constructed by partition of Δ .

Further suppose that we partition Δ sufficiently fine so that the distance between any pair of vertices of small simplices constructed by partition is sufficiently small. Let K be the set of small simplices constructed by partition of Δ , and $\mathbf{p} = (p_0, p_1, \dots, p_n)$ and $\mathbf{p}' = (p'_0, p'_1, \dots, p'_n)$ be vertices of a simplex of K . Denote the value of F at \mathbf{p} by $F(\mathbf{p})$. Let $\varphi(\mathbf{p})$ be a point in $F(\mathbf{p})$, and denote the i -th component of $\varphi(\mathbf{p})$ by φ_i , and so on. Since F has a closed graph, given sufficiently fine partition there exists δ such that if $|\mathbf{p} - \mathbf{p}'| < \delta$, then for $\varepsilon > 0$ $|\varphi(\mathbf{p}) - \varphi(\mathbf{p}')| < \varepsilon$ for any $\varphi(\mathbf{p}) \in F(\mathbf{p})$ and some $\varphi(\mathbf{p}') \in F(\mathbf{p}')$, or for some $\varphi(\mathbf{p}) \in F(\mathbf{p})$ and any $\varphi(\mathbf{p}') \in F(\mathbf{p}')$ ³.

Let \mathbf{p}^0 be a vertex of a small n -dimensional simplex of K which is labeled with 0 by the labelling method which will be explained below. We take a point $\varphi(\mathbf{p}) \in F(\mathbf{p})$ for all other vertices of this simplex so that $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p})| < \varepsilon$ is satisfied⁴.

It is important how to label the vertices contained in the faces of Δ . We label a vertex \mathbf{p} according to the following rule,

$$\text{If } p_k > \varphi_k \text{ or } p_k + \tau > \varphi_k, \text{ we label } \mathbf{p} \text{ with } k \text{ for } \varphi(\mathbf{p}) \in F(\mathbf{p}),$$

where τ is a positive number. If there are multiple k 's which satisfy this condition, we label \mathbf{p} conveniently for the conditions for Sperner's lemma to be satisfied. We do not randomly label the vertices.

For example, let \mathbf{p} be a point contained in an $n - 1$ -dimensional face of Δ such that $p_i = 0$ for one i among $0, 1, 2, \dots, n$ (the i -th component of its coordinates is 0). With $\tau > 0$, we have $\varphi_i > 0$ or $\varphi_i < \tau$ for each $\varphi(\mathbf{p}) \in F(\mathbf{p})$ ⁵. When

³Consider a sequence $(\mathbf{p}_m)_{m \geq 1}$ converging to \mathbf{p}' and a sequence $(\varphi(\mathbf{p}_m))_{m \geq 1}$ such that $\varphi(\mathbf{p}_m) \in F(\mathbf{p}_m)$ for each m , then closedness of the graph of F implies that $(\varphi(\mathbf{p}_m))_{m \geq 1}$ converges to a point in $F(\mathbf{p}')$. Similarly, consider a sequence $(\mathbf{p}'_m)_{m \geq 1}$ converging to \mathbf{p} and a sequence $(\varphi(\mathbf{p}'_m))_{m \geq 1}$ such that $\varphi(\mathbf{p}'_m) \in F(\mathbf{p}'_m)$ for each m , then closedness of the graph of F implies that $(\varphi(\mathbf{p}'_m))_{m \geq 1}$ converges to a point in $F(\mathbf{p})$.

⁴There may exist a case such that for any $\delta > 0$ we can not take a point $\varphi(\mathbf{p})$ for some vertex \mathbf{p} so that $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p})| < \varepsilon$ is satisfied. See Note at the end of this proof about such a case.

⁵In constructive mathematics for any real number x we can not prove that $x \geq 0$ or $x < 0$, that $x > 0$ or $x = 0$ or $x < 0$. But for any distinct real numbers x, y and z such that $x > z$ we can prove that $x > y$ or $y > z$.

$\varphi_i > 0$, from $\sum_{j=0}^n p_j = 1$, $\sum_{j=0}^n \varphi_j = 1$ and $p_i = 0$,

$$\sum_{j=0, j \neq i}^n p_j > \sum_{j=0, j \neq i}^n \varphi_j.$$

Then, for at least one j (denote it by k) we have $p_k > \varphi_k$, and we label \mathbf{p} with k , where k is one of the numbers which satisfy $p_k > \varphi_k$. Since $\varphi_i > p_i = 0$, i does not satisfy this condition. Assume $\varphi_i < \tau$. $p_i = 0$ implies $\sum_{j=0, j \neq i}^n p_j = 1$. Since

$\sum_{j=0, j \neq i}^n \varphi_j \leq 1$, we obtain

$$\sum_{j=0, j \neq i}^n p_j \geq \sum_{j=0, j \neq i}^n \varphi_j.$$

Then, for a positive number τ we have

$$\sum_{j=0, j \neq i}^n (p_j + \tau) > \sum_{j=0, j \neq i}^n \varphi_j.$$

There is at least one $j (\neq i)$ which satisfies $p_j + \tau > \varphi_j$. Denote it by k , and we label \mathbf{p} with k . k is one of the numbers other than i such that $p_k + \tau > \varphi_k$ is satisfied. i itself satisfies this condition ($p_i + \tau > \varphi_i$). But, since there is a number other than i which satisfies this condition, we can select a number other than i . We have proved that we can label the vertices contained in an $n - 1$ -dimensional face of Δ such that $p_i = 0$ for one i among $0, 1, 2, \dots, n$ with the numbers other than i . By similar procedures we can show that we can label the vertices contained in an $n - 2$ -dimensional face of Δ such that $p_i = 0$ for two i 's among $0, 1, 2, \dots, n$ with the numbers other than those i 's, and so on.

Consider the case where $p_i = p_{i+1} = 0$. We see that when $\varphi_i > 0$ or $\varphi_{i+1} > 0$,

$$\sum_{j=0, j \neq i, i+1}^n p_j > \sum_{j=0, j \neq i, i+1}^n \varphi_j,$$

and so for at least one j (denote it by k) we have $p_k > \varphi_k$, and we label \mathbf{p} with k . On the other hand, when $\varphi_i < \tau$ and $\varphi_{i+1} < \tau$, we have

$$\sum_{j=0, j \neq i, i+1}^n p_j \geq \sum_{j=0, j \neq i, i+1}^n \varphi_j.$$

Then, for a positive number τ we have

$$\sum_{j=0, j \neq i, i+1}^n (p_j + \tau) > \sum_{j=0, j \neq i, i+1}^n \varphi_j.$$

Thus, there is at least one $j (\neq i, i+1)$ which satisfies $p_j + \tau > \varphi_j$. Denote it by k , and we label \mathbf{p} with k .

Next consider the case where $p_i = 0$ for all i other than n . If for some i $\varphi_i > 0$, then we have $p_n > \varphi_n$, and label \mathbf{p} with n . On the other hand, if $\varphi_j < \tau$ for all $j \neq n$, then we obtain $p_n \geq \varphi_n$. It implies $p_n + \tau > \varphi_n$. Thus, we can label \mathbf{p} with n .

Therefore, the conditions for Sperner's lemma are satisfied, and there exists an odd number of fully labeled simplices in K .

2. Let $\mathbf{p}^0, \mathbf{p}^1, \dots$ and \mathbf{p}^n be the vertices of a fully labeled simplex. We name these vertices so that $\mathbf{p}^0, \mathbf{p}^1, \dots, \mathbf{p}^n$ are labeled, respectively, with $0, 1, \dots, n$. The values of F at these vertices are $F(\mathbf{p}^0), F(\mathbf{p}^1), \dots$ and $F(\mathbf{p}^n)$. Take points $\varphi(\mathbf{p}^0), \varphi(\mathbf{p}^1), \dots$ and $\varphi(\mathbf{p}^n)$ such that $\varphi(\mathbf{p}^0) \in F(\mathbf{p}^0), \varphi(\mathbf{p}^1) \in F(\mathbf{p}^1), \dots$ and $\varphi(\mathbf{p}^n) \in F(\mathbf{p}^n)$. The i -th components of \mathbf{p}^0 and $\varphi(\mathbf{p}^0)$ are denoted by \mathbf{p}_i^0 and $\varphi(\mathbf{p}^0)_i$, and so on.

By our assumption in (1) of this proof when the distance between \mathbf{p}^0 and \mathbf{p}^1 ($|\mathbf{p}^0 - \mathbf{p}^1|$) is smaller than δ , the distance between $\varphi(\mathbf{p}^0)$ and $\varphi(\mathbf{p}^1)$ ($|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p}^1)|$) is smaller than ε . We can make δ satisfying $\delta < \varepsilon^6$. Suppose $\tau > 0$. About \mathbf{p}^0 , from the labeling rules we have $\mathbf{p}_1^0 + \tau > \varphi(\mathbf{p}^0)_1$. About \mathbf{p}^1 , also from the labeling rules we have $\mathbf{p}_1^1 + \tau > \varphi(\mathbf{p}^1)_1$ which implies $\mathbf{p}_1^1 > \varphi(\mathbf{p}^1)_1 - \tau$. $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p}^1)| < \varepsilon$ means $\varphi(\mathbf{p}^1)_1 > \varphi(\mathbf{p}^0)_1 - \varepsilon$. On the other hand, $|\mathbf{p}^0 - \mathbf{p}^1| < \delta$ means $\mathbf{p}_1^0 > \mathbf{p}_1^1 - \delta$. Thus, from

$$\mathbf{p}_1^0 > \mathbf{p}_1^1 - \delta, \mathbf{p}_1^1 > \varphi(\mathbf{p}^1)_1 - \tau, \varphi(\mathbf{p}^1)_1 > \varphi(\mathbf{p}^0)_1 - \varepsilon$$

we obtain

$$\mathbf{p}_1^0 > \varphi(\mathbf{p}^0)_1 - \delta - \varepsilon - \tau > \varphi(\mathbf{p}^0)_1 - 2\varepsilon - \tau$$

By similar arguments, for each i other than 0 ,

$$\mathbf{p}_i^0 > \varphi(\mathbf{p}^0)_i - 2\varepsilon - \tau. \quad (3.1)$$

For $i = 0$ we have $\mathbf{p}_0^0 + \tau > \varphi(\mathbf{p}^0)_0$. Then,

$$\mathbf{p}_0^0 > \varphi(\mathbf{p}^0)_0 - \tau \quad (3.2)$$

⁶For example, for $\delta < 1$ and $\varepsilon < 1$, if when $|\mathbf{p}^0 - \mathbf{p}^1| < \delta$ we have $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p}^1)| < \varepsilon$, then we have $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p}^1)| < \varepsilon$ also when $|\mathbf{p}^0 - \mathbf{p}^1| < \delta\varepsilon < \varepsilon$.

Adding (3.1) and (3.2) side by side except for some i (denote it by k) other than 0,

$$\sum_{j=0, j \neq k}^n \mathbf{p}_j^0 > \sum_{j=0, j \neq k}^n \varphi(\mathbf{p}^0)_j - 2(n-1)\varepsilon - n\tau.$$

From $\sum_{j=0}^n \mathbf{p}_j^0 = 1$, $\sum_{j=0}^n \varphi(\mathbf{p}^0)_j = 1$ we have $1 - \mathbf{p}_k^0 > 1 - \varphi(\mathbf{p}^0)_k - 2(n-1)\varepsilon - n\tau$, which is rewritten as

$$\mathbf{p}_k^0 < \varphi(\mathbf{p}^0)_k + 2(n-1)\varepsilon + n\tau.$$

Since (3.1) implies $\mathbf{p}_k^0 > \varphi(\mathbf{p}^0)_k - 2\varepsilon - \tau$, we have

$$\varphi(\mathbf{p}^0)_k - 2\varepsilon - \tau < \mathbf{p}_k^0 < \varphi(\mathbf{p}^0)_k + 2(n-1)\varepsilon + n\tau.$$

Thus,

$$|\mathbf{p}_k^0 - \varphi(\mathbf{p}^0)_k| < 2(n-1)\varepsilon + n\tau \quad (3.3)$$

is derived. On the other hand, adding (3.1) from 1 to n yields

$$\sum_{j=1}^n \mathbf{p}_j^0 > \sum_{j=1}^n \varphi(\mathbf{p}^0)_j - 2n\varepsilon - n\tau.$$

From $\sum_{j=0}^n \mathbf{p}_j^0 = 1$, $\sum_{j=0}^n \varphi(\mathbf{p}^0)_j = 1$ we have

$$1 - \mathbf{p}_0^0 > 1 - \varphi(\mathbf{p}^0)_0 - 2n\varepsilon - n\tau. \quad (3.4)$$

Then, from (3.2) and (3.4) we get

$$|\mathbf{p}_0^0 - \varphi(\mathbf{p}^0)_0| < 2n\varepsilon + n\tau. \quad (3.5)$$

Since n is finite, and ε and τ are positive numbers which may be arbitrarily small, $2n\varepsilon + n\tau$ and $2(n-1)\varepsilon + n\tau$ may also be arbitrarily small. Let replace $2n\varepsilon + n\tau$ by ε , from (3.3) and (3.5) we obtain the following result,

$$|\mathbf{p}_i^0 - \varphi(\mathbf{p}^0)_i| < \varepsilon \text{ for all } i.$$

Redefining $(n+1)\varepsilon$ as ε , we get

$$|\mathbf{p}^0 - \varphi(\mathbf{p}^0)| < \varepsilon. \quad (3.6)$$

Since $\varphi(\mathbf{p}^0) \in F(\mathbf{p}^0)$, \mathbf{p}^0 is an approximate fixed point of F .

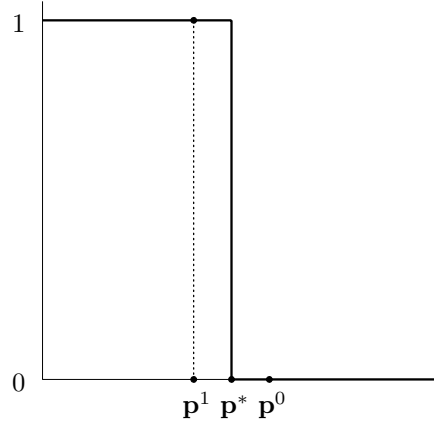


Figure 3: A multi-function in 1-dimensional case

Note

There may exist a case such that for any $\delta > 0$ we can not take a point $\varphi(\mathbf{p})$ for some vertex \mathbf{p} so that $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p})| < \varepsilon$ is satisfied. An example in a 1-dimensional case is a multi-function from $[0, 1]$ to $[0, 1]$ depicted in Figure 3. The coordinates of the points 0 and 1 are, respectively, $(0, 1)$ and $(1, 0)$. Coordinates for other points in $[0, 1]$ are similar. Even if $|\mathbf{p}^0 - \mathbf{p}^1| < \delta$ for any $\delta > 0$, $|\varphi(\mathbf{p}^0) - \varphi(\mathbf{p}^1)| > 0$. \mathbf{p}^0 and \mathbf{p}^1 are, respectively, numbered with 0 and 1. In such a case we must consider further partition of a simplex $[\mathbf{p}^0, \mathbf{p}^1]$ and take a limit when $\delta \rightarrow 0$. At the limit of vertices of a fully labeled simplex \mathbf{p}^* there are points $\varphi^1(\mathbf{p}^*) \in F(\mathbf{p}^*)$ and $\varphi^2(\mathbf{p}^*) \in F(\mathbf{p}^*)$ such that

$$\mathbf{p}_0^* > \varphi^1(\mathbf{p}^*)_0 - \tau \text{ and } \mathbf{p}_1^* > \varphi^2(\mathbf{p}^*)_1 - \tau.$$

Since $\mathbf{p}_0^* + \mathbf{p}_1^* = 1$ and $\varphi^2(\mathbf{p}^*)_0 + \varphi^2(\mathbf{p}^*)_1 = 1$, the latter implies

$$\mathbf{p}_0^* < \varphi^2(\mathbf{p}^*)_0 + \tau.$$

Thus,

$$\varphi^1(\mathbf{p}^*)_0 - \tau < \mathbf{p}_0^* < \varphi^2(\mathbf{p}^*)_0 + \tau.$$

Define a point in $F(\mathbf{p}^*)$ by

$$\varphi^*(\mathbf{p}^*) = \alpha \varphi^1(\mathbf{p}^*) + (1 - \alpha) \varphi^2(\mathbf{p}^*), \quad 0 \leq \alpha \leq 1.$$

By the convexity of $F(\mathbf{p}^*)$, $\varphi^*(\mathbf{p}^*) \in F(\mathbf{p}^*)$. Let

$$\alpha = \frac{\varphi^2(\mathbf{p}^*)_0 + \tau - \mathbf{p}_0^*}{[\varphi^2(\mathbf{p}^*)_0 + \tau - \mathbf{p}_0^*] + [\mathbf{p}_0^* - \varphi^1(\mathbf{p}^*)_0 + \tau]} = \frac{\varphi^2(\mathbf{p}^*)_0 + \tau - \mathbf{p}_0^*}{\varphi^2(\mathbf{p}^*)_0 - \varphi^1(\mathbf{p}^*)_0 + 2\tau},$$

and

$$1 - \alpha = \frac{\mathbf{p}_0^* - \varphi^1(\mathbf{p}^*)_0 + \tau}{\varphi^2(\mathbf{p}^*)_0 - \varphi^1(\mathbf{p}^*)_0 + 2\tau}.$$

Then,

$$\varphi^*(\mathbf{p}^*)_0 = \frac{\varphi^1(\mathbf{p}^*)_0(\tau - \mathbf{p}_0^*) + \varphi^2(\mathbf{p}^*)_0(\tau + \mathbf{p}_0^*)}{\varphi^2(\mathbf{p}^*)_0 - \varphi^1(\mathbf{p}^*)_0 + 2\tau}.$$

And so we have

$$\mathbf{p}_0^* - \varphi^*(\mathbf{p}^*)_0 = \frac{\tau[2\mathbf{p}_0^* - \varphi^1(\mathbf{p}^*)_0 - \varphi^2(\mathbf{p}^*)_0]}{\varphi^2(\mathbf{p}^*)_0 - \varphi^1(\mathbf{p}^*)_0 + 2\tau}.$$

Since τ may be arbitrarily small, for any $\varepsilon > 0$ we obtain

$$|\mathbf{p}_0^* - \varphi^*(\mathbf{p}^*)_0| < \varepsilon.$$

Similarly

$$|\mathbf{p}_1^* - \varphi^*(\mathbf{p}^*)_1| < \varepsilon$$

is derived. Therefore, \mathbf{p}^* is an approximate fixed point.

A case of higher dimension is similar. ■

We have completed the proof of a constructive version of Kakutani's fixed point theorem.

4. Approximate mini-max theorem

In this section we derive an approximate version of the mini-max theorem from the constructive version of Kakutani's fixed point theorem. Consider a two person zero-sum game. There are two players A and B . Player A has m alternative pure strategies, and the set of his pure strategies is denoted by $S_A = \{a_1, a_2, \dots, a_m\}$. Player B has n alternative pure strategies, and the set of his pure strategies is denoted by $S_B = \{b_1, b_2, \dots, b_n\}$. The payoff of player A when a combination of players' strategies is (a_i, b_j) is denoted by $M(a_i, b_j)$. Since we consider a zero-sum game, the payoff of player B is equal to $-M(a_i, b_j)$. Let p_i be a probability that A chooses his strategy a_i , and q_j be a probability that B chooses his strategy b_j . A mixed strategy of A is represented by a probability

distribution over S_A , and is denoted by $\mathbf{p} = (p_1, p_2, \dots, p_m)$ with $\sum_{i=1}^m p_i = 1$. Similarly,

a mixed strategy of B is denoted by $\mathbf{q} = (q_1, q_2, \dots, q_n)$ with $\sum_{j=1}^n q_j = 1$. A combination

of mixed strategies (\mathbf{p}, \mathbf{q}) is called a *profile*. The expected payoff of player A at a profile (\mathbf{p}, \mathbf{q}) is written as follows,

$$M(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^m \sum_{j=1}^n p_i M(a_i, b_j) q_j.$$

We assume that $M(a_i, b_j)$ is finite. Then, since $M(\mathbf{p}, \mathbf{q})$ is linear with respect to probability distributions over the sets of pure strategies of players, it is a uniformly continuous function. The expected payoff of A when he chooses a pure strategy a_i and B chooses a mixed strategy \mathbf{q} is $M(a_i, \mathbf{q}) = \sum_{j=1}^n M(a_i, b_j)q_j$, and that when A chooses a mixed

strategy \mathbf{p} and B chooses a pure strategy b_j is $M(\mathbf{p}, b_j) = \sum_{i=1}^m p_i M(a_i, b_j)$. The set of all mixed strategies of A is denoted by P , and that of B is denoted by Q . P is an $m - 1$ -dimensional simplex, and Q is an $n - 1$ -dimensional simplex. We call $v_A(\mathbf{p}) = \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q})$ the *guaranteed payoff* of A at \mathbf{p} . “inf” denotes the infimum (or greatest lower bound)⁷. Further, we define v_A^* as follows,

$$v_A^* = \sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q})$$

“sup” denotes the supremum (or least upper bound)⁸.

Similarly, we call $v_B(\mathbf{q}) = \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q})$ the guaranteed payoff of player B at \mathbf{q} , and define v_B^* as follows,

$$v_B^* = \inf_{\mathbf{q}} \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q}).$$

For a fixed \mathbf{p} we have $\inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) \leq M(\mathbf{p}, \mathbf{q})$ for all \mathbf{q} , and so

$$\sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) \leq \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q}) \text{ for all } \mathbf{q}$$

holds. Then, we obtain $\sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) \leq \inf_{\mathbf{q}} \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q})$. This is rewritten as

$$v_A^* \leq v_B^*. \quad (4.1)$$

⁷In constructive mathematics we can not prove the existence of the minimum of a uniformly continuous function on a compact set. Instead we can prove the existence of the infimum. Let y be the infimum of $M(\mathbf{p}, \mathbf{q})$. Then, y satisfies the following condition.

For all (\mathbf{p}, \mathbf{q}) , $y \leq M(\mathbf{p}, \mathbf{q})$, and for any $\varepsilon > 0$ and some (\mathbf{p}, \mathbf{q}) , $y > M(\mathbf{p}, \mathbf{q}) - \varepsilon$.

⁸In constructive mathematics we can not prove also the existence of the maximum of a uniformly continuous function on a compact set. Instead we can prove the existence of the supremum. Let z be the supremum of $\inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q})$. Then z satisfies the following condition.

For all (\mathbf{p}, \mathbf{q}) , $z \geq \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q})$, and for any $\varepsilon > 0$ and some (\mathbf{p}, \mathbf{q}) , $z < \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) + \varepsilon$.

Now, let ε be an arbitrary positive number, we define the following sets.

$$\Gamma_A(\mathbf{q}) = \{\mathbf{p} \in P \mid M(\mathbf{p}, \mathbf{q}) > M(\mathbf{p}', \mathbf{q}) - \varepsilon \text{ for all } \mathbf{p}' \in P\},$$

and

$$\Gamma_B(\mathbf{p}) = \{\mathbf{q} \in Q \mid M(\mathbf{p}, \mathbf{q}) < M(\mathbf{p}, \mathbf{q}') + \varepsilon \text{ for all } \mathbf{q}' \in Q\}.$$

Define a multi-function from $P \times Q$ to the set of inhabited subsets of $P \times Q$ by

$$\Theta(\mathbf{p}, \mathbf{q}) = (\Gamma_A(\mathbf{q}), \Gamma_B(\mathbf{p})).$$

Let us check that this multi-function satisfies the conditions for the constructive version of Kakutani's fixed point theorem.

1. Since $P \times Q$ is the product of two simplices, it is compact and convex. And since there are $m + n - 2$ independent vectors in $P \times Q$, $P \times Q$ is homeomorphic to an $m + n - 2$ -dimensional simplex.
2. $\Theta(\mathbf{p}, \mathbf{q})$ is a multi-function from $P \times Q$ to the set of all inhabited subsets of $P \times Q$.
3. We show convexity of $\Theta(\mathbf{p}, \mathbf{q})$. It is sufficient to show convexity of $\Gamma_A(\mathbf{q})$. Suppose that $\mathbf{p}^1 \in \Gamma_A(\mathbf{q})$ and $\mathbf{p}^2 \in \Gamma_A(\mathbf{q})$. Then,

$$M(\mathbf{p}^1, \mathbf{q}) > M(\mathbf{p}', \mathbf{q}) - \varepsilon \text{ for all } \mathbf{p}' \in P$$

and

$$M(\mathbf{p}^2, \mathbf{q}) > M(\mathbf{p}', \mathbf{q}) - \varepsilon \text{ for all } \mathbf{p}' \in P$$

hold. Since $M(\mathbf{p}, \mathbf{q})$ is linear with respect to probability distributions over the sets of pure strategies of players, for $0 \leq \lambda \leq 1$ we have

$$\lambda M(\mathbf{p}^1, \mathbf{q}) + (1 - \lambda)M(\mathbf{p}^2, \mathbf{q}) = M(\lambda \mathbf{p}^1 + (1 - \lambda)\mathbf{p}^2, \mathbf{q}) > M(\mathbf{p}', \mathbf{q}) - \varepsilon \text{ for all } \mathbf{p}' \in P.$$

Thus, we obtain $\lambda \mathbf{p}^1 + (1 - \lambda)\mathbf{p}^2 \in \Gamma_A(\mathbf{q})$, and $\Gamma_A(\mathbf{q})$ is convex. Convexity of $\Gamma_B(\mathbf{p})$ is similarly proved.

4. We show that $\Theta(\mathbf{p}, \mathbf{q})$ uniformly has a closed graph. Let \mathbf{p}'' be a mixed strategy of player A , \mathbf{q}'' be a mixed strategy of player B and $\mathbf{p} \in \Gamma_A(\mathbf{q})$. Uniform continuity of $M(\mathbf{p}, \mathbf{q})$ implies that, for a positive number ε , we can select $\delta > 0$ so that when $|(\mathbf{p}'', \mathbf{q}'') - (\mathbf{p}, \mathbf{q})| < \delta$ and $|(\mathbf{p}', \mathbf{q}'') - (\mathbf{p}', \mathbf{q})| < \delta$, we have $|M(\mathbf{p}'', \mathbf{q}'') - M(\mathbf{p}, \mathbf{q})| < \varepsilon$ and $|M(\mathbf{p}', \mathbf{q}'') - M(\mathbf{p}', \mathbf{q})| < \varepsilon$. Since $M(\mathbf{p}, \mathbf{q}) > M(\mathbf{p}', \mathbf{q}) - \varepsilon$ for all $\mathbf{p}' \in P$, we have

$$M(\mathbf{p}'', \mathbf{q}'') > M(\mathbf{p}, \mathbf{q}) - \varepsilon > M(\mathbf{p}', \mathbf{q}) - 2\varepsilon > M(\mathbf{p}', \mathbf{q}'') - 3\varepsilon \text{ for all } \mathbf{p}'.$$

Thus, $\mathbf{p}'' \in \Gamma_A(\mathbf{q}'')$. About $\Gamma_B(\mathbf{p})$ we can show a similar result. We have completed a proof that $\Theta(\mathbf{p}, \mathbf{q})$ uniformly has a closed graph.

Therefore, the conditions for the constructive version of Kakutani's fixed point theorem are satisfied by $\Theta(\mathbf{p}, \mathbf{q})$, and it has an approximate fixed point. Let denote one of the approximate fixed points by $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$. Then, for any $\frac{\varepsilon}{2} > 0$

$$M(\mathbf{p}', \tilde{\mathbf{q}}) - \frac{\varepsilon}{2} < M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) < M(\tilde{\mathbf{p}}, \mathbf{q}') + \frac{\varepsilon}{2} \text{ for all } (\mathbf{p}', \mathbf{q}')$$

holds. This means

$$\sup_{\mathbf{p}} M(\mathbf{p}, \tilde{\mathbf{q}}) - \frac{\varepsilon}{2} < M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) < \inf_{\mathbf{q}} M(\tilde{\mathbf{p}}, \mathbf{q}) + \frac{\varepsilon}{2}. \quad (4.2)$$

Since

$$\sup_{\mathbf{p}} M(\mathbf{p}, \tilde{\mathbf{q}}) \geq \inf_{\mathbf{q}} \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q}) = v_B^*, \quad \inf_{\mathbf{q}} M(\tilde{\mathbf{p}}, \mathbf{q}) \leq \sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) = v_A^*,$$

(4.2) implies

$$v_B^* - \frac{\varepsilon}{2} < M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) < v_A^* + \frac{\varepsilon}{2}. \quad (4.3)$$

With (4.1) and (4.3) we obtain

$$|v_A^* - v_B^*| < \varepsilon.$$

This v_A^* or v_B^* is the *value of the game*. v_A^* and v_B^* are approximated by $M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$. Summarizing the results,

Theorem 4.1. The value of a two person zero-sum game is determined within the range of ε , and it is approximated by $M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$. We can constructively find an approximate fixed point of a multi-function, and hence we can constructively find the strategies of players which realize the approximate value of the game.

Since $\varepsilon > 0$ is arbitrary, we obtain the result that $v_A^* = v_B^*$. But we can not constructively find the strategies of players which realize $v_A^* = v_B^*$.

5. Concluding Remarks

In this paper we have presented a proof of the existence of an approximate fixed point for any compact and convex valued multi-function from an n -dimensional simplex to the set of its inhabited subsets directly by Sperner's lemma and its application to a proof of the approximate mini-max theorem for zero-sum games from the viewpoint of constructive mathematics. We are studying some related problems such as the existence of an approximate equilibrium in an competitive economy with multi-valued demand and supply functions, a constructive version of the Fan-Glicksberg fixed point theorem for multi-functions in a locally convex space and its application to a proof of the existence of an approximate Nash equilibrium in a strategic game with continuous strategies and quasi-concave payoff functions.

Appendix

Proof of Sperner's lemma

We prove this lemma by induction about the dimension of Δ . When $n = 0$, we have only one point with the number 0. It is the unique 0-dimensional simplex. Therefore the lemma is trivial. When $n = 1$, a partitioned 1-dimensional simplex is a segmented line. The endpoints of the line are labeled distinctly, with 0 and 1. Hence in moving from endpoint 0 to endpoint 1 the labeling must switch an odd number of times, that is, an odd number of edges labeled with 0 and 1 may be located in this way.

Next consider the case of 2 dimension. Assume that we have partitioned a 2-dimensional simplex (triangle) Δ as explained above. Consider the face of Δ labeled with 0 and 1⁹. It is the base of the triangle in Figure 4. Now we introduce a dual graph that has its nodes in each small triangle of K plus one extra node outside the face of Δ labeled with 0 and 1 (putting a dot in each small triangle, and one dot outside Δ). We define edges of the graph that connect two nodes if they share a side labeled with 0 and 1. See Figure 4. White circles are nodes of the graph, and thick lines are its edges. Since from the result of 1-dimensional case there are an odd number of faces of K labeled with 0 and 1 contained in the face of Δ labeled with 0 and 1, there are an odd number of edges which connect the outside node and inside nodes. Thus, the outside node has odd degree. Since by the Handshaking lemma there are an even number of nodes which have odd degree, we have at least one node inside the triangle which has odd degree. Each node of our graph except for the outside node is contained in one of small triangles of K . Therefore, if a small triangle of K has one face labeled with 0 and 1, the degree of the node in that triangle is 1; if a small triangle of K has two such faces, the degree of the node in that triangle is 2, and if a small triangle of K has no such face, the degree of the node in that triangle is 0. Thus, if the degree of a node is odd, it must be 1, and then the small triangle which contains this node is labeled with 0, 1 and 2 (fully labeled). In Figure 4 triangles which contain one of the nodes A, B, C are fully labeled triangles.

Now assume that the theorem holds for dimensions up to $n - 1$. Assume that we have partitioned an n -dimensional simplex Δ . Consider the fully labeled face of Δ which is a fully labeled $n - 1$ -dimensional simplex. Again we introduce a dual graph that has its nodes in small n -dimensional simplices of K plus one extra node outside the fully labeled face of Δ (putting a dot in each small n -dimensional simplex, and one dot outside Δ). We define the edges of the graph that connect two nodes if they share a face labeled with 0, 1, \dots , $n - 1$. Since from the result of $n - 1$ -dimensional case there are an odd number of fully labeled faces of small simplices of K contained in the $n - 1$ -dimensional fully labeled face of Δ , there are an odd number of edges which connect the outside node and inside nodes. Thus, the outside node has odd degree. Since, by the Handshaking lemma there are an even number of nodes which have odd degree, we have at least one node inside the simplex which has odd degree. Each node of our graph except for the

⁹We call edges of triangle Δ *faces* to distinguish between them and edges of a dual graph which we will consider later.

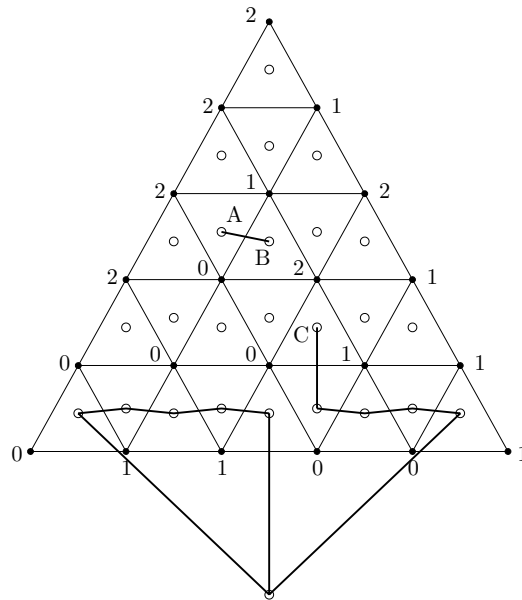


Figure 4: Sperner's lemma

outside node are contained in one of small n -dimensional simplices of K . Therefore, if a small simplex of K has one fully labeled face, the degree of the node in that simplex is 1; if a small simplex of K has two such faces, the degree of the node in that simplex is 2, and if a small simplex of K has no such face, the degree of the node in that simplex is 0. Thus, if the degree of a node is odd, it must be 1, and then the small simplex which contains this node is fully labeled.

If the number (label) of a vertex other than vertices labeled with $0, 1, \dots, n - 1$ of an n -dimensional simplex which contains a fully labeled $n - 1$ -dimensional face is n , then this n -dimensional simplex has one such face, and this simplex is a fully labeled n -dimensional simplex. On the other hand, if the number of that vertex is other than n , then the n -dimensional simplex has two such faces.

We have completed the proof of Sperner's lemma. ■

References

- [1] E. Bishop and D. Bridges, *Constructive Analysis*, Springer, 1985.
- [2] D. Bridges and F. Richman, *Varieties of Constructive Mathematics*, Cambridge University Press, 1987.
- [3] D. Bridges and L. Vîță, *Techniques of Constructive Mathematics*, Springer, 2006.

- [4] D. van Dalen, “Brouwer’s ε -fixed point from Sperner’s lemma”, *Theoretical Computer Science*, <http://dx.doi.org/10.1016/j.tcs.2011.04.002>, in press 2011.
- [5] R. B. Kellogg, T. Y. Li and J. Yorke, “A constructive proof of Brouwer fixed-point theorem and computational results”, *SIAM Journal on Numerical Analysis*, 13:473–483, 1976.
- [6] F. E. Su, “Rental harmony: Sperner’s lemma for fair division”, *American Mathematical Monthly*, 106:930–942, 1999.
- [7] W. Veldman, “Brouwer’s approximate fixed point theorem is equivalent to Brouwer’s fan theorem”, in *Logicism, Intuitionism and Formalism*, edited by Lindström, S., Palmgren, E., Segerberg, K. and Stoltenberg-Hansen, Springer, 2009.